Large Chiral Diffeomorphisms on Riemann Surfaces and $\mathcal{W}$-algebras

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Abstract

The diffeomorphism action lifted on truncated (chiral) Taylor expansion of a complex scalar field over a Riemann surface is presented in the paper under the name of large diffeomorphisms. After an heuristic approach, we show how a linear truncation in the Taylor expansion can generate an algebra of symmetry characterized by some structure functions. Such a linear truncation is explicitly realized by introducing the notion of Forsyth frame over the Riemann surface with the help of a conformally covariant algebraic differential equation. The large chiral diffeomorphism action is then implemented through a B.R.S. formulation (for a given order of truncation) leading to a more algebraic set up. In this context the ghost fields behave as holomorphically covariant jets. Subsequently, the link with the so called $\mathcal{W}$-algebras is made explicit once the ghost parameters are turned from jets into tensorial ghost ones. We give a general solution with the help of the structure functions pertaining to all the possible truncations lower or equal to the given order. This provides another contribution to the relationship between KdV flows and $\mathcal{W}$-diffeomorphims.

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1 Introduction

The notion of symmetry gives a structure to spacetime (or configuration space) and/or internal spaces of the model under consideration, in the sense that the former is closely related to a geometrical setup.

At the infinitesimal level, the concept of algebra turns out to be very useful. It generally gives rise to a solution when the linearity is fulfilled. In the non linear situation, however, we have sometimes to explore further behind the first infinitesimal transformation step.

The development of non linear sciences has been supported these needs [1], and many basic physical systems were described by non linear extensions of algebras. This is the case for integrable systems, two dimensional conformal models (with application to Strings, Gravity or Solid State Physics, see e.g. [2]).

Particular interest has been devoted to the so-called $\mathcal{W}$-algebras [3] which come out from different principles [4, 5, 6] and the question of their geometric origin remains still unclear or unsatisfactory despite the various attempts given by [5, 7, 8, 9, 10, 11].

In particular, the relationship between a conformally covariant s-th order differential equation

$$\left(\partial_{(s)} + a^{(s)}_{(2)}(z,\overline{z})\partial_{(s-2)} + \cdots + a^{(s)}_{(s)}(z,\overline{z})\right) f(z,\overline{z}) = 0 \ ,$$

over a generic Riemann surface and some of the so-called $\mathcal{W}$-algebra has been well established since fifteen years by [6]. The differential equations can be thought of as equations of motion for some matter fields. The former can be also derived as vanishing covariant derivative condition [12], and in some extent, more general $\mathcal{W}$-algebra as the Bershadsky one [13], can be related to a system of conformally covariant coupled differential equations.

Focussing here only on one s-th order differential equation, one may either work directly with the solutions or with uniformizing coordinates considered as ratios of linear independent solutions [14, 8]. It turns out to be a matter of taste of working either with homogeneous coordinates or inhomogeneous one on $\mathbb{C}P^{s-1}$. However, in his textbook [14], Forsyth uses rather inhomogeneous coordinates in order to get some differential invariants. The inhomogeneous coordinates have to satisfy a $s+1$-th order equations. Accordingly, our point of view proposed in [15] traced back in the literature, and found out an unexpected origin in old past [14] related to some algebraic type differential equations, once covariantly formulated over a generic Riemann surface. Indeed the definition of general well-defined differential operators over a (two dimensional) Riemann Surface requires some care [16], and puts forward a deep insight on the links between covariance required by physical considerations and projective geometry. Related studies of projectively invariant differential operators as well differential invariants can be found in [17].

In [15] the notion of Laguerre-Forsyth frames were promoted. In order to avoid a possible confusion with the named Laguerre-Forsyth form of a conformally covariant differential equation [8], in the present paper, we adopt the name Forsyth frames. We have simply in mind the ideas of, firstly, pursuing further ahead the method given by Forsyth in [14], and secondly, dealing with scalar coordinates considered as solutions of generalized Beltrami equations (see e.g. [8] or [18] appendix C2) just about the approach given in [8, 10]. Our motivation for using the inhomogeneous coordinates rests on the fact that they seem to be more natural for constructing projective invariants. Projective action of $SL(s,\mathbb{C})$ on $\mathbb{C}P^{s-1}$. In general [19], the smooth coefficients in the above s-th order covariant differential equation have been proven to be projectively invariant. Inhomogeneous coordinates are the local coordinates for projective curves in $\mathbb{C}P^{s-1}$ on which there is a symplectic structure related to the Gelfand-Dickey brackets [20, 19]. Moreover, inhomogeneous coordinates offer the possibility to work with scalar fields instead of densities.

In [15], these Forsyth frames were thus made of coordinate scalar fields on some finite dimensional target space and constructed from solutions of a finite order holomorphically covariant linear differential equation over a Riemann surface of the above type. At the quantum level, these scalar fields surprisingly gained already at the one-loop level a non commutative character. This phenomenon gives a quantum origin to a non commutative structure on the target space. The non-commutativity came out by anomaly cancellation as a nonlocal effect.

The basic novelties of these Forsyth frames lie in their non trivial properties under differentiations, which allow to expand beyond the first order the holomorphic reparametrization process still maintaining the algebra closure property. Thus investigate this type of extended algebra appears as a necessity, since this "new" structure encapsulates the (general) covariance laws.

In this paper, the construction of Forsyth frames is proposed in great details and it strongly relies on a symmetry principle. The latter will be systematically studied in the algebraic BRS language more suitable for a possible field theoretical treatment of the model.

The paper is organized as follows. Section 2 goes over some motivations for this extended notion of symmetry, beyond linearity. It requires the use of higher order derivatives and the closure of the algebra obliges to introduce new fields which will play the role of structure functions for the (non linear) reduced symmetry algebra. Section 3 is devoted to the definition of the Forsyth frames and to a deep study of their properties. In Section 4, a convenient BRS approach is presented for the algebra of symmetry. It will be useful for improving in particular the covariance laws which come out from these properties. Section 5 treats the covariance under holomorphic change charts of the algebra elements. Non tensorial structures (jet) come out, in particular under the form of jet-BRS ghosts which are harmful for physical considerations since covariant quantities are required. A rather tricky link with tensorial ghosts is presented as a change of basis of generators. Section 6 gives a way (using all the differential properties of the Forsyth frames, including the ones of the sub-frames encapsulated into the maximal one) to decompose into tensors the jet-BRS ghosts. The process is obviously defined up to a tensorial rescaling. Some detailed examples are given and illustrate the striking property of cancelling out the effects of all the sub-frames, in favor to a standard presentation of the $\mathcal{W}$-algebra structure. Exploiting some known results in the literature [10] allows to clarify the algorithm. We conclude in section 7.

2 Motivations

The issue of finding the most general expression of a spacetime symmetry can find an answer in the concept of generalized frames, perhaps rather of that of prolongated frames whose local expression is obtained by successive derivations [21]. In order to consider such objects, let us first think of the Fock space of some smooth complex scalar field Z defined over a given Riemann surface Σ, endowed with a complex analytic (holomorphic) structure; this requires the use of local complex coordinates $(z, \overline{z})$. Smoothness of the complex scalar field $Z = Z(z, \overline{z})$ is understood with respect to the differential structure on Σ with which the complex structure is subordinated to. From the locality principle, recall that the Fock space for Z is generated by all the z and $\overline{z}$ derivatives of any order considered as independent monomials.

Consider now the infinitesimal action of smooth diffeomorphisms on Σ which is usually expressed by means of the Lie derivative $L_\xi Z = (\xi(z,\overline{z})\partial + \overline{\xi}(z,\overline{z})\overline{\partial})Z$.

With respect to the complex structure we shall be concerned with the so-called "chiral" diffeomorphisms acting on the complex scalar field Z which consist in separating the Lie derivative according to z-derivative, so that $Z \longrightarrow Z + \xi\partial Z$; there is the complex conjugate expression as well. Denoting in the Fock space the various z-derivatives of order ℓ by $\partial_{(\ell)}Z$, $\ell = 1, 2, \ldots$ one wants to consider a fully chiral local variation for the complex scalar field Z (going thus beyond the first order) of the following type

$$(\delta Z)(z,\overline{z}) = \sum_{\ell\geq 1} \gamma^{(\ell)}(z,\overline{z})\, \partial_{(\ell)} Z(z,\overline{z}), \tag{2.1}$$

together with the complex conjugate expression, and into which all the z-derivatives appear.

Hence, constructing a local field theory over $\mathbb{C}$ amounts to working a priori on local functionals in the various z-derivatives of Z. So to speak, the physical model is built over the infinite z-frames which is locally represented by $(Z(z,\overline{z}), \partial Z(z,\overline{z}), \partial_{(2)} Z(z,\overline{z}), \dots)$, which reproduces the "chiral" Taylor expansion of the field Z (i.e. only with respect to the z-coordinate) at the point $(z,\overline{z})$ of Σ, in other words, the infinite jet of Z.

What is called large chiral diffeomorphisms in the paper, is the lifted action of usual local chiral diffeomorphisms on $\mathbb{C}$ to the infinite jet space $J^\infty(\mathbb{C},\mathbb{C})$, i.e. on the z-Taylor expansion. The former are viewed as transformations acting on the complex scalar field Z itself and they require, on the one hand, infinitely many local parameters $\gamma^{(\ell)}$ of conformal type $(-\ell, 0)$ which generalize vector fields of type $(-1,0)$, and, on the other hand any higher order derivatives of Z.

Accordingly, it is called a large chiral diffeomorphism symmetry the invariance of some observables or functional on $J^\infty(\mathbb{C},\mathbb{C})$ under the transformation (2.1). Translating this problem of symmetry on the space of local functionals in the infinite jet of the scalar field Z,

$$\delta Z(z,\overline{z}) = \left(\sum_{\ell \geq 1} \int_{\mathbb{C}} d\overline{w} \wedge dw \; \gamma^{(\ell)}(w,\overline{w}) \mathcal{W}_{(\ell)}(w,\overline{w}) \right) Z(z,\overline{z}), \tag{2.2}$$

amounts to introducing local Ward operators associated to the local parameters $\gamma^{(\ell)}$

$$\mathcal{W}_{(\ell)}(z,\overline{z}) = \partial_{(\ell)} Z(z,\overline{z}) \frac{\delta}{\delta Z(z,\overline{z})}, \qquad \text{for } \ell \geq 1, \tag{2.3}$$

which generate an infinite dimensional Lie algebra. But note that the only Lie sub-algebra is for $\ell = 1$ of which the bracket (as tensorial product of distributions) writes

$$\begin{aligned} \Big[\mathcal{W}_{(1)}(z,\overline{z}), \mathcal{W}_{(1)}(w,\overline{w}) \Big] &= \mathcal{W}_{(1)}(z,\overline{z}) \partial_w \delta(w-z) - \mathcal{W}_{(1)}(w,\overline{w}) \partial_z \delta(z-w) \\ &= \mathcal{W}_{(1)}(w,\overline{w}) \partial_w \delta(w-z) - \mathcal{W}_{(1)}(z,\overline{z}) \partial_z \delta(z-w) \end{aligned} \tag{2.4}$$

and translates by duality the Lie algebra structure of vector fields to the functional level, namely, if $W_1(\xi) = \Big\langle \mathcal{W}_1 \,|\, \xi \Big\rangle$ then

$$\begin{aligned} \Big[W_1(\xi), W_1(\eta) \Big] &= \Big\langle \Big[\mathcal{W}_{(1)}(z,\overline{z}), \mathcal{W}_{(1)}(w,\overline{w}) \Big] \Big| \, \xi(z,\overline{z}) \eta(w,\overline{w}) \Big\rangle \\ &= \Big\langle \mathcal{W}_1 \Big| \, \eta \partial \xi - \xi \partial \eta \Big\rangle = \Big\langle \mathcal{W}_1 \Big| \, [\eta, \xi] \Big\rangle = W_1([\eta,\xi]) \end{aligned} \tag{2.5}$$

and thus reproduces the bracket between the parameters of usual conformal transformations. (The pairing $\langle \, | \, \rangle$ stands for the functional evaluation of distributions). In full generality, the brackets for $k, \ell \geq 1$ read

$$\begin{aligned} \Big[\mathcal{W}_{(k)}(z,\overline{z}), \mathcal{W}_{(\ell)}(w,\overline{w}) \Big] &= \sum_{m=1}^{\ell} \binom{\ell}{m} \partial_w^m \delta(w-z) \mathcal{W}_{(k+\ell-m)}(w,\overline{w}) \\ &\quad - \sum_{m=1}^{k} \binom{k}{m} \partial_z^m \delta(z-w) \mathcal{W}_{(k+\ell-m)}(z,\overline{z}) \end{aligned} \tag{2.6}$$

and close onto subspaces generated by $\{\mathcal{W}_{(u)}\}_{u=\min(k,\ell)}^{k+\ell-1}$ and leads to introducing higher order generators at each step. Note that the bracket (2.6) fulfills the Jacobi identity. Moreover for arbitrary k and $\ell = 1$ one obtains by duality

$$\begin{aligned} \Big\langle \Big[\mathcal{W}_{(k)}(z,\overline{z}), \mathcal{W}_{(1)}(w,\overline{w}) \Big] \Big| \, \xi^{(k)}(z,\overline{z}) \eta^{(1)}(w,\overline{w}) \Big\rangle &= \Big\langle \mathcal{W}_{(k)} \Big| \, \eta^{(1)} \partial \xi^{(k)} - k \, \partial \eta^{(1)} \xi^{(k)} \Big\rangle \\ &\quad - \sum_{m=2}^{k \geq 2} \binom{k}{m} \Big\langle \mathcal{W}_{(k-m+1)} \Big| \, \xi^{(k)} \partial^m \eta^{(1)} \Big\rangle \end{aligned} \tag{2.7}$$

where the first smearing bracket on the right hand side shows that the conformal transformations $\mathcal{W}_{(1)}$ preserve the $\mathcal{W}_{(k)}$ transformations up to lower orders and defines a covariant bracket $[\eta^{(1)},\xi^{(k)}]^{(k)} = \eta^{(1)}\partial\xi^{(k)} - k\,\partial\eta^{(1)}\xi^{(k)}$ showing that the parameter $\xi^{(k)}$ carries a conformal weight $(-k,0)$.

The closure onto a finite dimensional Lie sub-algebra for $\ell > 1$, can be obtained by a truncation in the z-derivatives of $Z(z,\overline{z})$ at the some finite order, say $s-1$, $(s \geq 2)$. Setting for z-derivatives of order greater than $s-1$, namely for $m \geq s$, the following linear combinations

$$\partial_{(m)}Z(z,\overline{z}) = \sum_{\ell=1}^{s-1}\mathcal{R}^{(\ell)}_{(m)}(z,\overline{z})\,\partial_{(\ell)}Z(z,\overline{z}) \Longrightarrow \mathcal{W}_{(m)}(z,\overline{z}) = \sum_{\ell=1}^{s-1}\mathcal{R}^{(\ell)}_{(m)}(z,\overline{z})\,\mathcal{W}_{(\ell)}(z,\overline{z}) \tag{2.8}$$

where the finite summation run over $\ell = 1,\ldots,s-1$, thus the immediate consequence is that the bracket (2.6) reduces to

$$\begin{aligned}\Big[\mathcal{W}_{(k)}(z,\overline{z}),\mathcal{W}_{(\ell)}(w,\overline{w})\Big] = \sum_{u=1}^{s-1}\Bigg\{&\sum_{m=1}^{\ell}\binom{\ell}{m}\,\partial_w^m\delta(w-z)\mathcal{R}^{(u)}_{(k+\ell-m)}(w,\overline{w})\mathcal{W}_{(u)}(w,\overline{w})\\ &-\sum_{m=1}^{k}\binom{k}{m}\,\partial_z^m\delta(z-w)\mathcal{R}^{(u)}_{(k+\ell-m)}(z,\overline{z})\mathcal{W}_{(u)}(z,\overline{z})\Bigg\},\end{aligned} \tag{2.9}$$

and closes onto the generators $\{\mathcal{W}_{(u)}\}_{u=1}^{s-1}$. However, the truncation spoils the Jacobi identity due to the restriction of the order to the range $\ell = 1,\ldots,s-1$. The Jacobi identity (which guarantees the associativity of the Lie algebra) will be restored by modifying the generators $\{\mathcal{W}_{(u)}\}_{u=1}^{s-1}$ in such a way to take into account the reduction coefficients $\mathcal{R}^{(u)}_{(v)}(z,\overline{z})$ introduced in (2.8). These coefficients will play the role of "structure functions" for the finite dimensional Lie algebra generated by the modified generators $\{\widetilde{\mathcal{W}}_{(u)}\}_{u=1}^{s-1}$. Owing to (2.8), this is achieved by computing, for $k \geq s$ and for $\ell = 1,\ldots,s-1$, the difference

$$\begin{aligned}\Big[\mathcal{W}_{(k)}(z,\overline{z}),\mathcal{W}_{(\ell)}(w,\overline{w})\Big] - \sum_{p=1}^{s-1}\mathcal{R}^{(p)}_{(k)}(z,\overline{z})\Big[\mathcal{W}_{(p)}(z,\overline{z}),\mathcal{W}_{(\ell)}(w,\overline{w})\Big] &=:\\ \sum_{u=1}^{s-1}\Big[\mathcal{R}^{(u)}_{(k)}(z,\overline{z}),\mathcal{F}_{(\ell)}\Big(\mathcal{R}(w,\overline{w}),\frac{\delta}{\delta\mathcal{R}(w,\overline{w})}\Big)\Big]&\mathcal{W}_{(u)}(z,\overline{z}),\end{aligned} \tag{2.10}$$

where $\mathcal{F}_{(\ell)}\Big(\mathcal{R}(w,\overline{w}),\frac{\delta}{\delta\mathcal{R}(w,\overline{w})}\Big)$ is a functional differential polynomial in the $\mathcal{R}$'s and insures the modification of the generator, for each $\ell = 1,\ldots,s-1$, according to

$$\mathcal{W}_{(\ell)}(w,\overline{w}) \;\longrightarrow\; \widetilde{\mathcal{W}}_{(\ell)}(w,\overline{w}) = \partial_{(\ell)}Z(w,\overline{w})\frac{\delta}{\delta Z(w,\overline{w})} + \mathcal{F}_{(\ell)}\Big(\mathcal{R}(w,\overline{w}),\frac{\delta}{\delta\mathcal{R}(w,\overline{w})}\Big) \tag{2.11}$$

in view to fulfill the Jacobi identity. Furthermore, of course one has for $\ell = 1,\ldots,s-1$

$$\mathcal{R}^{(\ell)}_{(k)}(z,\overline{z}) = \begin{cases}\delta^\ell_k & \text{if } 1 \leq k \leq s-1,\\ \mathcal{R}^{(\ell)}_{(k)}(z,\overline{z}) & \text{if } k \geq s,\end{cases} \tag{2.12}$$

and therefore the functional operator $\mathcal{F}_{(\ell)}$ must contain functional derivatives with respect to the structure functions $\mathcal{R}^{(p)}_{(k)}$ for $k \geq s$ only. Thus inserting twice the brackets (2.9) into the right hand side of (2.10), a direct comparison with the left hand side of (2.10) amounts, on the one hand, to the vanishing of the coefficient terms of the $\mathcal{W}_{(u)}(w,\overline{w})$'s. This gives rise to some compatibility conditions that must be fulfilled by the structure functions, namely,

$$\mathcal{R}^{(u)}_{(k+n)}(w,\overline{w}) = \sum_{p=1}^{s-1}\sum_{j=0}^{n}\binom{n}{j}\partial_{(j)}\mathcal{R}^{(p)}_{(k)}(w,\overline{w})\mathcal{R}^{(u)}_{(p+n-j)}(w,\overline{w}), \qquad \text{for } n = 0,\ldots,\ell \text{ and } k \geq s. \tag{2.13}$$

If k is taken to be lower or equal to $s-1$ then (2.13) restricts to $\mathcal{R}^{(u)}_{(k+n)} = \mathcal{R}^{(u)}_{(k+n)}$, since $\mathcal{R}^{(p)}_{(k)} = \delta^p_k$.

On the other hand, the coefficient term of $\mathcal{W}_{(u)}(z,\overline{z})$ provides the functional differential operator

$$
\begin{aligned}
\mathcal{F}_{(\ell)}\left(\mathcal{R}(w,\overline{w}),\frac{\delta}{\delta\mathcal{R}(w,\overline{w})}\right) &= \sum_{i\geq s}\sum_{j=1}^{s-1}\left\{\sum_{m=0}^{i}\binom{i}{m}(-1)^m\partial_{(m)}\left(\mathcal{R}^{(j)}_{(i+\ell-m)}(w,\overline{w})\frac{\delta}{\delta\mathcal{R}^{(j)}_{(i)}(w,\overline{w})}\right)\right.\\
&\left. -\sum_{p=1}^{s-1}\sum_{q=0}^{p}\binom{p}{q}(-1)^q\partial_{(q)}\left(\mathcal{R}^{(p)}_{(i)}(w,\overline{w})\mathcal{R}^{(j)}_{(p+\ell-q)}(w,\overline{w})\frac{\delta}{\delta\mathcal{R}^{(j)}_{(i)}(w,\overline{w})}\right)\right\}. \qquad (2.14)
\end{aligned}
$$

Therefore, in addition to the scalar field Z, the structure functions $\mathcal{R}$'s come as new fields to be taken into account in the theory. Their variation is obtained to be

$$
\begin{aligned}
\delta\mathcal{R}^{(p)}_{(n)}(z,\overline{z}) &= \left(\int_{\mathbb{C}} d\overline{w}\wedge dw\ \left[\sum_{\ell=1}^{s-1}\gamma^{(\ell)}(w,\overline{w})+\sum_{u\geq s}\gamma^{(u)}(w,\overline{w})\mathcal{R}^{(\ell)}_{(u)}(w,\overline{w})\right]\widetilde{\mathcal{W}}_{(\ell)}(w,\overline{w})\right)\mathcal{R}^{(p)}_{(n)}(z,\overline{z})\\
&= \sum_{\ell=1}^{s-1}\left[\sum_{m=0}^{n}\binom{n}{m}\partial_{(m)}\left(\gamma^{(\ell)}+\sum_{u\geq s}\gamma^{(u)}\mathcal{R}^{(\ell)}_{(u)}\right)\mathcal{R}^{(p)}_{(n+\ell-m)}\right.\\
&\left.\quad -\sum_{j=1}^{s-1}\mathcal{R}^{(j)}_{(n)}\sum_{q=0}^{j}\binom{j}{q}\partial_{(q)}\left(\gamma^{(\ell)}+\sum_{u\geq s}\gamma^{(u)}\mathcal{R}^{(\ell)}_{(u)}\right)\mathcal{R}^{(p)}_{(j+\ell-q)}\right](z,\overline{z}),
\end{aligned} \qquad (2.15)
$$

while the variation (2.2) for Z rewrites

$$
\delta Z(z,\overline{z}) = \left(\int_{\mathbb{C}} d\overline{w}\wedge dw\ \left[\sum_{\ell=1}^{s-1}\gamma^{(\ell)}(w,\overline{w})+\sum_{u\geq s}\gamma^{(u)}(w,\overline{w})\mathcal{R}^{(\ell)}_{(u)}(w,\overline{w})\right]\widetilde{\mathcal{W}}_{(\ell)}(w,\overline{w})\right)Z(z,\overline{z}). \qquad (2.16)
$$

To this change $\mathcal{W}_{(\ell)} \longrightarrow \widetilde{\mathcal{W}}_{(\ell)}$ (for $\ell = 1,\ldots,s-1$) of generators there corresponds a reduction from an infinite number to a finite number of local parameters

$$
\gamma^{(m)} \longrightarrow \Gamma^{(\ell)} = \sum_{m\geq 1}\gamma^{(m)}\mathcal{R}^{(\ell)}_{(m)}, \qquad \ell = 1,\ldots,s-1 \qquad (2.17)
$$

as suggested by both the variations (2.15) and (2.16). The $s-1$ local parameters $\Gamma^{(\ell)}$ secure the fact that the $s-1$ generators $\widetilde{\mathcal{W}}_{(\ell)}$ fulfill indeed the algebra (2.9). In short, this leads to a reduction of the symmetry algebra, and (2.1) reduces to the variation

$$
(\delta Z)(z,\overline{z}) = \sum_{\ell=1}^{s-1}\Gamma^{(\ell)}(z,\overline{z})\,\partial_{(\ell)}Z(z,\overline{z}). \qquad (2.18)
$$

By duality the following brackets $[\ ,\ ]^{(u)}$ corresponding to the generators $\widetilde{\mathcal{W}}_{(u)}$ are found to be

$$
[\eta^{(\ell)},\xi^{(k)}]^{(u)} = \sum_{m=0}^{\ell-1}\binom{\ell}{m}\mathcal{R}^{(u)}_{(k+m)}\,\eta^{(\ell)}\partial_{(\ell-m)}\xi^{(k)} - \sum_{m=0}^{k-1}\binom{k}{m}\mathcal{R}^{(u)}_{(\ell+m)}\,\xi^{(k)}\partial_{(k-m)}\eta^{(\ell)}. \qquad (2.19)
$$

These brackets are involved in the defining Poisson brackets for $\mathcal{W}$-algebras [3, 22, 23]. This leads to the

Conclusion 2.1 *The realization of large diffeomorphism algebra Eq.(2.6) requires the definition of frames which verify the truncation property Eq.(2.8) which realizes a derivative order reduction (D.O.R). So the structure functions $\mathcal{R}^{(u)}_{(v)}(z,\overline{z})$ uniquely define the properties of the algebra.*

The problem we are after is twofold. First, due to the presence of higher order derivatives which carry a non tensorial nature (jets), one wants to perform the construction in a well defined way, in the sense that this local symmetry has indeed a global meaning over the Riemann surface Σ. That is, constructing a field theory over the coframes $J^\infty(\Sigma,\mathbb{C})$. Second, find the appropriate generators for the symmetry algebra, (jets or tensors), which give rise to covariant quantities over the Riemann surface these quantities being constructed from covariant differential operators, covariant in the sense to be holomorphically well defined on Σ. This would correspond to a change of generators $\{\widetilde{\mathcal{W}}_{(u)}\}_{u=1}^{s-1}$ in order to get a presentation of the Poisson $\mathcal{W}$-algebras which are no longer Lie algebras. This means in particular that the lower orders in the brackets (2.7) should not be present any more [23].

This second goal requires in fact to work with a finite number of $s-1$ complex scalar fields Z instead of one only. Therefore, one is led to consider the jet space $J^\infty(\Sigma,\mathbb{C}^{s-1})$ on which local diffeomorphisms of $\mathbb{C}^{s-1}$ stabilizing the target point $(Z^{(1)},\ldots,Z^{(s-1)})$ are lifted by jet composition and act linearly. Presently, a truncation in the order of the jet can be implemented by means of relations given by some PDE's. The simplest ones are given by a linear PDE which yields an algebraic relation between jet coordinates. This is what will be developed in the next section.

3 The Forsyth frames

Over a Riemann surface Σ, let us introduce the algebraic PDE of fixed order s with smooth coefficients and defined by

$$L_s f(z,\bar z)=0, \qquad \text{with } L_s=\sum_{j=0}^{s} a^{(s)}_{(s-j)}(z,\bar z)\partial_{(j)}, \qquad \text{where } a^{(s)}_{(0)}(z,\bar z)=1, \text{ and } a^{(s)}_{(1)}(z,\bar z)=0, \quad (3.1)$$

When $\bar z$ is viewed to play the role of a parameter the PDE is considered as an ODE in the independent variable z and the function f is the unknown. It thus introduces a chiral splitting between the complex coordinates. Around any point of Σ this ODE admits s linearly independent *local* solutions $f^{(R)}$, $R=1,\ldots,s$ on a small enough neighborhood of any point. Actually, any solution turns out to be a scalar density under holomorphic changes of charts $(U,z)\to(\widehat U,w(z))$ with conformal weight $\frac{1-s}{2}$ in order to have a well defined covariance on the Riemann surface Σ which yields

$$L_s(w,\bar w)=(w')^{-\frac{1+s}{2}}L_s(z,\bar z)\,(w')^{\frac{1-s}{2}} \qquad \text{on } U\cap\widehat U\neq\emptyset. \quad (3.2)$$

Recall that $L_s f$ has conformal weight $\frac{1+s}{2}$. For an overview see e.g. [24] and references therein. Equation (3.1) can be recast as a first-order differential operator if the jet of order $s-1$ of any solution f is considered as the variable. One has

$$\Big(\partial + A^{(s)}(z,\bar z)\Big)\begin{pmatrix} f(z,\bar z)\\ \partial f(z,\bar z)\\ \vdots\\ \partial_{(r-1)}f(z,\bar z)\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\\ L_s f(z,\bar z)\end{pmatrix}=0 \quad (3.3)$$

where the $s\times s$ matrix

$$A^{(s)}(z,\bar z)=\begin{pmatrix} 0 & -1 & 0 & \ldots & 0\\ 0 & 0 & -1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \ldots & 0 & -1\\ a^{(s)}_{(s)}(z,\bar z) & a^{(s)}_{(s-1)}(z,\bar z) & \ldots & a^{(s)}_{(2)}(z,\bar z) & a^{(s)}_{(1)}(z,\bar z)\end{pmatrix} \quad (3.4)$$

has entries

$$[A^{(s)}(z,\bar{z})]_{lm} = \begin{cases} -1 & \text{for } m = l+1, \\ a^{(s)}_{(s-m+1)}(z,\bar{z}) & \text{for } l = s, \text{ with } a^{(s)}_{(1)}(z,\bar{z}) \equiv 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Moreover, each of them in the last row carries a z lower index content of covariant order $s-m+1$. But in account of (3.3), $A^{(s)}$ is expected to carry a covariant index z as the derivative ∂_z.

This allows one to associate to the ODE a system of s identical equations by introducing s unknowns $f^{(R)}$, $R = 1, \ldots, s$. So it is a system of the same ODE over $\Sigma \times \mathbb{C}^s$. Note that any linear change in the local linearly independent solutions $\widetilde{f}^{(R)}(z,\bar{z}) = A^{(R)}_{(S)} f^{(S)}(z,\bar{z})$ over $U \subset \Sigma$ preserves (3.1), $A \in GL(s,\mathbb{C})$. For the time being, the matrix A does not depend on the local complex coordinates $(z,\bar{z})$ on Σ, but this issue should be tackled as the gauging of the largest symmetry group of the ODE (3.1).

However, adapting Gunning [25] for a well defined ODE over Σ according to (3.2) if one selects s linearly independent local solutions of (3.1), on the one hand, $f^{(R)}(z,\bar{z})$ on the coordinate neighborhood $U \subset \Sigma$, and on the other hand, $\widehat{f}^{(R)}(w,\bar{w})$ on the coordinate neighborhood $\widehat{U} \subset \Sigma$, then on the overlapping of these coordinate neighborhoods one has, in full generality, the following gluing rule for solutions

$$\text{on } U \cap \widehat{U} \neq \emptyset, \qquad \widehat{f}^{(R)}(w,\bar{w}) = (w')^{-\frac{1-s}{2}}\, T^{(R)}_{(S)} f^{(S)}(z,\bar{z}) \tag{3.6}$$

where the unique constant matrix T tuns out to be a 1-cocycle of the chosen coordinate covering on Σ with values in $GL(s,\mathbb{C})$. To prove (3.6), for the given linearly independent s solutions $\widehat{f}^{(R)}(w,\bar{w})$ in the open set $\widehat{U} \subset \Sigma$, let us introduce the functions $g^{(R)}(z,\bar{z})$ on $U \cap \widehat{U} \subset U$ defined by

$$g^{(R)}(z,\bar{z}) = (w')^{\frac{1-s}{2}} \widehat{f}^{(R)}(w,\bar{w}) \qquad \text{on } U \cap \widehat{U} \neq \emptyset. \tag{3.7}$$

Upon using the covariance law (3.2) it is readily seen that the s functions $g^{(R)}(z,\bar{z})$ are linearly independent solutions of (3.1) over $U \cap \widehat{U} \subset U$. Accordingly, for any other s linearly independent $f^{(R)}(z,\bar{z})$ of (3.1) in the open set U, the functions $g^{(R)}(z,\bar{z})$ are unique linear combinations of the functions $f^{(R)}(z,\bar{z})$, that is $g^{(R)}(z,\bar{z}) = T^{(R)}_{(S)}\, f^{(S)}(z,\bar{z})$ a fact which demonstrates (3.6).

According to [14], (see also [8]) one may define *locally* $s-1$ smooth scalar fields over a neighborhood of any point of Σ as quotients of $s-1$ local solutions by a preferred one which does not vanish on a neighborhood of a given point, say on U,

$$Z^{(R)}(z,\bar{z}) = \frac{f^{(R+1)}(z,\bar{z})}{f^{(1)}(z,\bar{z})}, \qquad R = 1, \cdots, s-1 \tag{3.8}$$

where the functions $f^{(R+1)}$ and $f^{(1)}$ belong to the same set of linearly independent solutions over U. By virtue of (3.6) one checks that

$$\text{on } U \cap \widehat{U} \neq \emptyset, \qquad \widehat{Z}^{(R)}(w,\bar{w}) = \frac{T^{(R+1)}_{(S+1)} Z^{(S)}(z,\bar{z}) + T^{(R+1)}_{(1)}}{T^{(1)}_{(S+1)} Z^{(S)}(z,\bar{z}) + T^{(1)}_{(1)}}, \tag{3.9}$$

a transformation law which shows that the local scalar fields Z have to be transformed in a homographic way in accordance with the Zucchini's point of view [7] on $\mathcal{W}$-algebras.

Moreover, note that any linear change in the linearly independent solutions $f^{(R)}$ on U induces a homographic transformation in the $Z^{(R)}$ on U as

$$\widetilde{Z}^{(R)}(z,\bar{z}) = \frac{A^{(R+1)}_{(S+1)} Z^{(S)}(z,\bar{z}) + A^{(R+1)}_{(1)}}{A^{(1)}_{(S+1)} Z^{(S)}(z,\bar{z}) + A^{(1)}_{(1)}}. \tag{3.10}$$

The question of gauging whether or not the matrix $A \in GL(s, \mathbb{C})$ into $A(z, \overline{z})$, in other words, render local the above transformation law should be also tackled in the sequel.

This is the point where now one has to decide if $Z^{(R)}$ is a genuine scalar field on Σ or not, namely keeping T as general as possible or reduce it to the identity. If one chooses the latter what would be the meaning of $T = I$ for the space of solutions of the ODE (3.1) ? A possible answer would be that there exists basis of solutions which is globally defined on Σ according to (3.6) with $T = I$. But there is a more precise statement which is the following. Consider

$$\text{on } \widehat{U}, \qquad \widehat{Z}^{(R)}(w, \overline{w}) = \frac{\widehat{f}^{(R+1)}(w, \overline{w})}{\widehat{f}^{(1)}(w, \overline{w})}, \qquad R = 1, \cdots, s-1 \tag{3.11}$$

and thus with (3.7) one can define on the overlapping the scalar functions

$$\text{on } U \cap \widehat{U} \neq \emptyset, \qquad \zeta^{(R)}(z, \overline{z}) = \frac{g^{(R+1)}(z, \overline{z})}{g^{(1)}(z, \overline{z})} = \widehat{Z}^{(R)}(w, \overline{w}), \qquad R = 1, \cdots, s-1 \tag{3.12}$$

which thus coincide with the scalar functions $\widehat{Z}^{(R)}$ on the intersection $U \cap \widehat{U} \neq \emptyset$. Since in the glueing rule given by (3.6) the matrix T only depends on the two coordinate neighborhoods U and $\widehat{U}$, let us perform the following linear change $\widetilde{f} = a\,T f$ of linearly independent solutions on the open set U, with given $a: \ U \to \mathbb{C}^*$. This yields a gauge transformation over U of the type (3.10) with $A = T$ so that

$$\text{on } U \cap \widehat{U} \neq \emptyset, \qquad \zeta^{(R)}(z, \overline{z}) = \widehat{Z}^{(R)}(w, \overline{w}) = \widetilde{Z}^{(R)}(z, \overline{z}), \qquad R = 1, \cdots, s-1. \tag{3.13}$$

Hence by these redefinitions through gauge transformations, one can construct $s-1$ scalar fields on Σ, still denoted by $Z^{(R)}$, each of those being a collection of scalar maps defined on the various coordinate neighborhoods satisfying (3.13) as matching rule.

Suppose now that a family of linear differential equations of the type (3.1) is indexed by the order $r \geq 2$. Accordingly, solutions must be labeled by the order r, and the above construction holds for each of the orders r. One may state the

Conjecture 3.1 *Around each point of the Riemann surface Σ and for each integer $r \geq 2$, there is a map, $\Sigma \to \mathbb{C}P^{r-1}$ which defines local inhomogeneous coordinates on $\mathbb{C}P^{r-1}$, collectively denoted by the vector valued in $\mathbb{C}^{r-1}$ smooth function on Σ*

$$\vec{Z}(z, \overline{z}|r) = (Z^{(1)}(z, \overline{z}|r), \ldots, Z^{(r-1)}(z, \overline{z}|r)), \tag{3.14}$$

where the $r-1$ components are given by

$$Z^{(R)}(z, \overline{z}|r) \ = \ \frac{f^{(R+1)}(z, \overline{z}|r)}{f^{(1)}(z, \overline{z}|r)}, \qquad R = 1, \cdots, r-1. \tag{3.15}$$

$\vec{Z}(z, \overline{z}|r)$ will be called a Forsyth frame. For a given a point on Σ, all the frames must be equivalent for all the physical points of view.

Returning to the general discussion with a given equation (3.1) of order s which introduces the truncation in the jet order, the following theorem comes as a by-product.

Theorem 3.1 *For any $m \geq 1$, given an order s of truncation dictated by a differential equation of the type (3.1), one has*

$$\partial_{(m)} \vec{Z}(z, \overline{z}) = \sum_{l=1}^{s-1} \mathcal{R}^{(l)}_{(m)}(z, \overline{z}) \partial_{(l)} \vec{Z}(z, \overline{z}), \quad \text{and} \quad \mathcal{R}^{(l)}_{(m)}(z, \overline{z}) = \delta^{(l)}_{(m)} \quad \text{if } 1 \leq m \leq s-1 \tag{3.16}$$

The decomposition is universal for all the inhomogeneous coordinates, $Z^{(R)}(z,\overline{z})$ in the sense that each $\mathcal{R}^{(l)}_{(m)}$ does not depend on the index (R) of the former. The $\mathcal{R}$'s correspond exactly to those heuristically introduced in (2.8) and are specific to the order s of truncation imposed by (3.1). However the vectorial character of $\vec{Z}$ gives rise to some additional compatibility conditions between themselves.

Proof 1 *The proof of Theorem 3.1 is trivial by direct computation. Indeed we can compute $\partial_s f^{(P+1)}(z,\overline{z})$ for all $P = 1,\dots,s-1$ in two different ways. The first one comes from the very definition Eq.(3.8) and the Leibniz rule*

$$\partial_{(s)} f^{(P+1)} = \partial_{(s)}\left(Z^{(P)} f^{(1)}\right) = \partial_{(s)} Z^{(P)} f^{(1)} + Z^{(P)}\partial_{(s)} f^{(1)} + \sum_{j=1}^{s-1}\binom{s}{j}\partial_{(j)} Z^{(P)}\partial_{(s-j)} f^{(1)} \qquad (3.17)$$

The second one comes from from the very definition Eq.(3.8) and the fact that both $f^{(P+1)}$ and $f^{(1)}$ are solutions of Eq.(3.1):

$$\begin{aligned}
\partial_{(s)} f^{(P+1)} &= -\sum_{j=0}^{s-1} a^{(s)}_{(s-j)}\partial_{(j)}\left(Z^{(P)} f^{(1)}\right) = -\sum_{j=0}^{s-1} a^{(s)}_{(s-j)}\sum_{m=0}^{j}\binom{j}{m}\partial_{(m)} Z^{(P)}\partial_{(j-m)} f^{(1)}\\
&= -Z^{(P)}\sum_{j=0}^{s-1} a^{(s)}_{(s-j)}\partial_{(j)} f^{(1)}(z,\overline{z}) - \sum_{j=1}^{s-1} a^{(s)}_{(s-j)}\sum_{m=1}^{j}\binom{j}{m}\partial_{(m)} Z^{(P)}\partial_{(j-m)} f^{(1)}\\
&= Z^{(P)}\partial_{(s)} f^{(1)} - \sum_{m=1}^{s-1}\partial_{(m)} Z^{(P)}\sum_{j=m}^{s-1} a^{(s)}_{(s-j)}\binom{j}{m}\partial_{(j-m)} f^{(1)} \qquad (3.18)
\end{aligned}$$

A direct comparison between Eqs.(3.17) and (3.18) entails

$$\partial_{(s)} Z^{(P)} = \frac{-1}{f^{(1)}}\sum_{m=1}^{s-1}\left(\binom{s}{m}\partial_{(s-m)} f^{(1)} + \sum_{j=m}^{s-1} a^{(s)}_{(s-j)}\binom{j}{m}\partial_{(j-m)} f^{(1)}\right)\partial_{(m)} Z^{(P)} \qquad (3.19)$$

The decomposition (3.16) combined with the non vanishing of the Wronskian determinant (3.21) yields

$$\mathcal{R}^{(m)}_{(s)}(z,\overline{z}) \equiv \frac{-1}{f^{(1)}(z,\overline{z})}\left(\binom{s}{m}\partial_{(s-m)} f^{(1)}(z,\overline{z}) + \sum_{j=m}^{s-2} a^{(s)}_{(s-j)}(z,\overline{z})\binom{j}{m}\partial_{(j-m)} f^{(1)}(z,\overline{z})\right) \qquad (3.20)$$

where $\mathcal{R}^{(m)}_{(s)}$ for $m = 1,\dots,s-1$ depend on the coefficients $a^{(s)}_{(s-j)}$ and $f^{(1)}$ and its z derivatives up to order $s-2$ since $a^{(s)}_{(1)} = 0$. It is readily seen that the decomposition does not depend on the index of the solution $f^{(P+1)}$. One can extend (3.20) to the case $m = 0$ since $f^{(1)}$ is solution of (3.1) by setting $\mathcal{R}^{(0)}_{(s)} \equiv 0$.

Let us introduce the Wronskian as a $(s-1)\times(s-1)$-matrix

$$\varpi(z,\overline{z}) = \left(\varpi^{(R)}_{(\ell)}(z,\overline{z})\right) = \begin{pmatrix} \partial Z^{(1)}(z,\overline{z}) & \cdots & \partial Z^{(s-1)}(z,\overline{z})\\ \vdots & \ddots & \vdots\\ \partial_{(s-1)} Z^{(1)}(z,\overline{z}) & \cdots & \partial_{(s-1)} Z^{(s-1)}(z,\overline{z}) \end{pmatrix}. \qquad (3.21)$$

Hence in the algebra of squared matrices of order $s-1$ the relationships (3.16) state that any z-derivative of the Wronskian ϖ can be decomposed as a product of a rectangular matrix with the functions $\mathcal{R}$ as entries by ϖ, in details,

$$\partial_{(m)}\varpi^{(R)}_{(\ell)} = \sum_{k=1}^{s-1}\mathcal{R}^{(k)}_{(m+\ell)}\varpi^{(R)}_{(k)}. \qquad (3.22)$$

In order to be the most general as possible, the Wronskian may be extended to a $m\times(s-1)$ rectangular matrix when higher derivatives $m\geq s$ of the $Z^{(R)}$'s are considered. In account of (3.16), the rectangular matrix of derivatives of Z up to order m can always be expressed in terms of the Wronskian matrix (3.21). We shall call this mechanism connected to the truncation heuristically introduced in (2.9) as a Derivative Order Reduction, or in shorthand D.O.R. Note also that thanks to (3.16) a straightforward computation gives

$$\mathcal{R}^{(s-1)}_{(s)}(z,\bar{z}) = \partial \ln\det\varpi(z,\bar{z}). \tag{3.23}$$

The preferred solution $f^{(1)}$ which crucially takes place in the computation of the $\mathcal{R}$'s plays a distinguished role in the construction as it has been already seen. In particular it infers a linear relationship for $f^{(1)}$ with $j=s-1$ in Eq.(3.20)

$$\mathcal{R}^{(s-1)}_{(s)}(z,\bar{z}) = -s\,\partial\ln f^{(1)}(z,\bar{z}) \tag{3.24}$$

which yields, on the one hand, together with (3.23)

$$f^{(1)}(z,\bar{z}) = (\det\varpi(z,\bar{z}))^{-1/s}, \tag{3.25}$$

and on the other hand,

$$\Longrightarrow \partial f^{(1)} = \mathcal{Q}_{(1)} f^{(1)}, \qquad \text{where } \mathcal{Q}_{(1)} = -\tfrac{1}{s}\mathcal{R}^{(s-1)}_{(s)} \tag{3.26}$$

and by successive z-derivatives one gets a recursive formula

$$\partial_{(n)} f^{(1)}(z,\bar{z}) = \mathcal{Q}_{(n)}(z,\bar{z}) f^{(1)}(z,\bar{z}), \quad \text{with } \mathcal{Q}_{(n)} = \partial\mathcal{Q}_{(n-1)} + \mathcal{Q}_{(n-1)}\mathcal{Q}_{(1)}, \quad \text{and } \mathcal{Q}_{(0)} = 1, \tag{3.27}$$

so that $\mathcal{Q}_{(n)}$ turns to be a differential polynomial in $\mathcal{Q}_{(1)}$ (i.e. in $\mathcal{R}^{(s-1)}_{(s)}$), namely $\mathcal{Q}_{(n)} = (\partial+\mathcal{Q}_{(1)})^{n-1}\mathcal{Q}_{(1)}$. Using (3.27) into (3.20) and eliminating $f^{(1)}$ allow to write a linear system in a Gauss form with respect to the a's coefficients

$$\mathcal{R}^{(j)}_{(s)} + \begin{pmatrix} s \\ j \end{pmatrix} \mathcal{Q}_{(s-j)} + \sum_{l=j}^{s-1} \begin{pmatrix} \ell \\ j \end{pmatrix} a^{(s)}_{(s-\ell)} \mathcal{Q}_{(\ell-j)} = 0, \qquad \text{for } j=0,\ldots,s-1, \tag{3.28}$$

which expresses the relationship between the $a^{(s)}$'s and the $\mathcal{R}_{(s)}$'s. This step is independent of $f^{(1)}$ provided that the $\mathcal{R}_{(s)}$'s are given (together with some compatibility conditions) and we will consider from now on and throughout the all paper that the degrees of freedom will be the $\mathcal{R}_{(s)}$'s. Hence, solving iteratively the system (3.28) with respect to the $a^{(s)}$'s one gets

$$\begin{cases} a^{(s)}_{(1)} = 0 & \text{for } j=s-1 \\ a^{(s)}_{(2)} = -\mathcal{R}^{(s-2)}_{(s)} - \begin{pmatrix} s \\ s-2 \end{pmatrix}\mathcal{Q}_{(2)} & \text{for } j=s-2 \\ a^{(s)}_{(3)} = -\mathcal{R}^{(s-3)}_{(s)} - \begin{pmatrix} s \\ s-3 \end{pmatrix}\mathcal{Q}_{(3)} - \begin{pmatrix} s-2 \\ s-3 \end{pmatrix} a^{(s)}_{(2)}\mathcal{Q}_{(1)} & \text{for } j=s-3 \\ \quad\vdots & \text{and so on up to } j=0. \end{cases} \tag{3.29}$$

This shows that to a given a DOR (3.16) there corresponds a holomorphically covariant differential equation of the type (3.1) whose smooth coefficients are expressed as differential polynomials in the structure function $\mathcal{R}^{(s-1)}_{(s)}$ and linearly with respect to the others.

Moreover one has the following property which is exactly the compatibility condition (2.13).

Properties 1 *For $p=1,\ldots,s-1$ and $n\geq s$*

$$\mathcal{R}^{(p)}_{(m+n)}(z,\bar{z}) = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \sum_{\ell=1}^{s-1} \partial_{(j)}\mathcal{R}^{(\ell)}_{(n)}(z,\bar{z})\mathcal{R}^{(p)}_{(m+\ell-j)}(z,\bar{z}) \tag{3.30}$$

where $1\leq l\leq s-1$; so, recursively all the $\mathcal{R}^{(l)}_{(s+m)}(z,\bar{z})\quad m>0$ coefficients can be derived from the basic $\mathcal{R}^{(l)}_{(s)}(z,\bar{z})$ ones.

In particular, for the case $m=1$, one has

$$\mathcal{R}^{(p)}_{(n+1)}(z,\overline{z}) = \partial\mathcal{R}^{(p)}_{(n)}(z,\overline{z}) + \sum_{\ell=1}^{s-1}\mathcal{R}^{(\ell)}_{(n)}(z,\overline{z})\mathcal{R}^{(p)}_{(\ell+1)}(z,\overline{z}), \tag{3.31}$$

an equation which will be useful for future applications.

The basic $\mathcal{R}^{(l)}_{(s)}$'s, namely the structure functions given in the introductory section, play a central role and it would be worthwhile to have some hints about their geometric nature. In order to be closer as possible to a differential geometric setting for our approach, let us proceed as follows. For the the z-jet of a fixed order, one has the following holomorphic glueing rules under the change $z \mapsto w = w(z)$

$$\partial_z^{k+1}Z(z,\overline{z}) = \begin{cases} w'(z)\,\partial_w Z(w,\overline{w}) & \text{if } k=0, \\ w^{(k+1)}(z)\partial_w Z(w,\overline{w}) + \displaystyle\sum_{\ell=1}^{k}\partial_w^{\ell+1}Z(w,\overline{w})\times & \\ \quad\times\displaystyle\sum_{r=\ell}^{k}\frac{k!}{(k-r)!}\,w^{(k-r+1)}(z)\left(\sum_{\substack{a_1+\cdots+ra_r=r\\ a_1+\cdots+a_r=\ell}}\left(\prod_{n=1}^{r}\frac{1}{a_n!}\left(\frac{w^{(n)}(z)}{n!}\right)^{a_n}\right)\right) & \text{if } k\geq 1, \end{cases} \tag{3.32}$$

where the last expression comes from the use of the Faà di Bruno formula for higher order chain rule of derivatives. For further calculations, one has more explicitly and under a more elegant form, for $k \geq 3$

$$\begin{aligned}(\partial_{(k+1)}Z)(z,\overline{z}) \;=\; & (w')^{k+1}(\partial_{(k+1)}Z)(w,\overline{w}) + \binom{k+1}{2}(w')^{k}\partial_z\ln w'\,(\partial_{(k)}Z)(w,\overline{w}) \\ & + \binom{k+1}{3}(w')^{k-1}\Big(\{w,z\} + \tfrac{3}{4}\,k(\partial_z\ln w')^2\Big)(\partial_{(k-1)}Z)(w,\overline{w}) \\ & + \binom{k+1}{4}(w')^{k-2}\Big(\partial_z\{w,z\} + 2(k-1)\{w,z\}\partial_z\ln w' + \binom{k}{2}(\partial_z\ln w')^3\Big)(\partial_{(k-2)}Z)(w,\overline{w}) \\ & + \text{lower order derivatives,}\end{aligned} \tag{3.33}$$

where $\{w,z\} = \partial_z^2\ln w' - \frac{1}{2}(\partial_z\ln w')^2 = \dfrac{w'''}{w'} - \frac{3}{2}\left(\dfrac{w''}{w'}\right)^2$ denotes the Schwarzian derivative.

Now for fixed s, one can obtain the geometric properties of the $s-1$ structure functions $\mathcal{R}$'s by solving the linear system (3.16) with respect to the $\mathcal{R}$'s by Cramer method, one gets the following Lie form associated to the PDEs (3.16) (see eg [26]) for $m = 1,\ldots,s-1$

$$\Phi^{(m)}(\vec{Z}_s) := (-1)^{s-1-m}\,\frac{\det(\partial\vec{Z},\partial_{(2)}\vec{Z},\ldots,\widehat{\partial_{(m)}\vec{Z}},\ldots,\partial_{(s-1)}\vec{Z},\partial_{(s)}\vec{Z})}{\det\varpi} = \mathcal{R}^{(m)}_{(s)}(z,\overline{z}), \tag{3.34}$$

where the symbol $\widehat{\ }$ means omission. This expression can simply be rewritten as

$$\mathcal{R}^{(m)}_{(s)} = \partial_{(s)}Z^{(R)}[\varpi^{-1}]^{(m)}_{(R)}. \tag{3.35}$$

For $s=2$, one has the obvious relations $\mathcal{R}^{(1)}_{(m)} = \partial_{(m)}Z/\partial_{(1)}Z$. According to an approach advocated by Vessiot to the Picard-Vessiot theory [26] (and references therein) one can construct, regarding the present case and by a repeated use of (3.33), a natural holomorphic bundle with a $(s-1)$-dimensional fiber with fiber coordinates $(u^{(1)},\ldots,u^{(s-1)})$. It is defined by the following holomorphic transition functions induced by the holomorphic change of chart $w = \varphi(z)$ on Σ

$$\begin{cases} w = \varphi(z) \\ U^{(s-1)} = \frac{1}{w'}\Big(u^{(s-1)} - \binom{s}{2}\partial \ln w'\Big) \qquad\qquad \text{affine bundle !} \\ U^{(s-2)} = \frac{1}{w'^2}\Big(u^{(s-2)} + \binom{s-1}{2}u^{(s-1)}\partial \ln w' - \binom{s}{3}\big(\{w,z\} + \frac{3}{4}\begin{pmatrix} s-1 \\ 1 \end{pmatrix}(\partial \ln w')^2\big)\Big) \\ U^{(s-3)} = \frac{1}{w'^3}\Big(u^{(s-3)} + \binom{s-2}{2}u^{(s-2)}\partial \ln w' + u^{(s-1)}\binom{s-1}{3}\big(\{w,z\} + \frac{3}{4}\begin{pmatrix} s-2 \\ 1 \end{pmatrix}(\partial \ln w')^2\big) \\ \qquad\qquad - \binom{s}{4}\big(\partial\{w,z\} + 2\binom{s-2}{1}\{w,z\}\partial \ln w' + \binom{s-1}{2}(\partial \ln w')^3\big)\Big) \\ \quad \vdots \\ U^{(1)} = \text{a very intricate expression depending on all the } u^{(i)}\text{'s} \end{cases} \tag{3.36}$$

where the transition laws become more and more involved. This bundle can be recast into a holomorphic natural bundle of geometric objects (but however with smooth sections $\mathcal{R}$ in accordance with locality) as fibered product over the Riemann surface Σ

$$\mathcal{F}_{\text{affine}} \times_\Sigma \mathcal{F},$$

where $\mathcal{F}_{\text{affine}}$ is the affine bundle and $\mathcal{F}$ is the bundle with very intricate remaining but important patching rules for the sequel. Having at our disposal some of the main glueing rules of the fundamental $\mathcal{R}$'s, it is possible to obtain the geometrical nature of some of the coefficients of (3.1). Indeed, in terms of the Wronskian $\mathcal{R}^{(s-1)}_{(s)} = \partial \ln \det \varpi$, one finds for the coefficient

$$a^{(s)}_{(2)} = \tfrac{s-1}{2}\left(\partial \mathcal{R}^{(s-1)}_{(s)} - \tfrac{1}{s}(\mathcal{R}^{(s-1)}_{(s)})^2\right) - \mathcal{R}^{(s-2)}_{(s)}, \tag{3.37}$$

while both $\mathcal{R}^{(s-1)}_{(s)}$ and $\mathcal{R}^{(s-2)}_{(s)}$ glue as smooth sections of the bundle defined by (3.36). After a direct computation

$$a^{(s)}_{(2)}(z,\bar z) = (w')^2 a^{(s)}_{(2)}(w,\bar w) + \tfrac{s(s^2-1)}{12}\left(\partial^2 \ln w' - \tfrac{1}{2}(\partial \ln w')^2\right) \tag{3.38}$$

which shows that $a^{(s)}_{(2)}$ is proportional to a projective connection as is well known, since the inhomogeneous term in the glueing rule is nothing but the Schwarzian derivative $\{w,z\} = \partial^2 \ln w' - \frac{1}{2}(\partial \ln w')^2$. The projective connection is constructed over the frame $\vec Z$ according to (3.37).

Remark 3.1 *For the case $s = 3$, one has $\mathcal{R}^{(2)}_{(3)} = \partial \ln \det \varpi$, and $\mathcal{R}^{(1)}_{(3)} = \det(\partial^3 \vec Z, \partial^2 \vec Z)/\det \varpi$. With $\mathcal{Q}_{(1)} = \frac{-1}{3}\mathcal{R}^{(2)}_{(3)}$, one readily gets*

$$\begin{aligned} a^{(3)}_{(2)} &= -\mathcal{R}^{(1)}_{(3)} - 3\mathcal{Q}_{(2)} = -\mathcal{R}^{(1)}_{(3)} - 3(\partial \mathcal{Q}_{(1)} + (\mathcal{Q}_{(1)})^2) \\ a^{(3)}_{(3)} &= \mathcal{R}^{(1)}_{(3)}\mathcal{Q}_{(1)} + 2(\mathcal{Q}_{(1)})^3 - \partial^2 \mathcal{Q}_{(1)} = \tfrac{1}{3}\left(\partial a^{(3)}_{(2)} + \partial \mathcal{R}^{(1)}_{(3)} + \tfrac{2}{3}\mathcal{R}^{(2)}_{(3)} a^{(3)}_{(2)} - \tfrac{1}{3}\mathcal{R}^{(1)}_{(3)}\mathcal{R}^{(2)}_{(3)}\right) \end{aligned} \tag{3.39}$$

which are exactly those coefficients obtained for the so-called $\mathcal{W}_3$-algebra [14, 27]. The last expression is given in terms of the projective connection and the structure functions only.

Remark 3.2 *The factorization property of the differential operator L_s of order s in terms of 1st order differential operators with nowhere vanishing coefficients can be obtained if and only if L_s is a non-oscillating operator, see [20] for some details. For more concreteness, let us illustrate this factorization property for $s = 2, 3$.*

1. *The $s=2$ case. Take $f^{(1)} = (\partial Z)^{-1/2} =: \lambda^{-1/2}$ as a nowhere vanishing particular solution of $L_2 f = 0$. One can write*

$$L_2 = \partial^2 + a_{(2)}^{(2)} = (\partial - b)(\partial + b) \quad \textit{with } a_{(2)}^{(2)} = \partial b - b^2, \quad \textit{and } b = -\partial \ln f^{(1)} =: -\mathcal{Q}_{(1)}, \tag{3.40}$$

One finds the expected factorization

$$L_2 = \big(\partial - \tfrac{1}{2}\partial \ln \partial\lambda\big)\big(\partial + \tfrac{1}{2}\partial \ln \partial\lambda\big), \qquad \textit{and} \quad a_{(2)}^{(2)} = \tfrac{1}{2}\partial^2 \ln \partial\lambda - \tfrac{1}{4}(\partial \ln \partial\lambda)^2. \tag{3.41}$$

2. *The $s=3$ case amounts to writing*

$$L_3 = (\partial - b_1 - b_2)(\partial + b_2)(\partial + b_1) \tag{3.42}$$

with $b_1 = -\partial \ln f^{(1)} = -\mathcal{Q}_{(1)}$ and a possible choice for b_2 is given by $b_2 = b_1 - \partial \ln \partial Z^{(1)}$ –it could be possible to choose $b_2 = b_1 - \partial \ln \partial Z^{(2)}$ since b_1 never vanishes. Then substituting into

$$\begin{aligned} a_{(2)}^{(3)} &= \partial(b_1 + b_2) - (b_1 + b_2)^2 + \partial b_1 + b_1 b_2 \\ a_{(3)}^{(3)} &= \partial^2 b_1 + \partial(b_1 b_2) - (b_1 + b_2)(\partial b_1 + b_1 b_2), \end{aligned}$$

one exactly recovers the expressions given above in (3.39) for the coefficients of L_3.

Still with a fixed given order s, it is possible to construct a connection-like object. With the Dolbeault decomposition of the de Rham differential $\mathrm{d} = \boldsymbol{\partial} + \overline{\boldsymbol{\partial}}$ let us define the flat connection (pure gauge)

$$\mathcal{J} = \mathrm{d}\varpi\, \varpi^{-1} = \boldsymbol{\partial}\varpi\, \varpi^{-1} + \overline{\boldsymbol{\partial}}\varpi\, \varpi^{-1} = \mathrm{d}z \mathcal{J}_z + \mathrm{d}\overline{z}\mathcal{J}_{\overline{z}}. \tag{3.43}$$

Obviously its curvature vanishes

$$\mathcal{F} = \mathrm{d}\mathcal{J} - \mathcal{J}^2 = 0 \quad \Longrightarrow \quad \overline{\partial}_{\overline{z}}\mathcal{J}_z - \partial_z \mathcal{J}_{\overline{z}} + [\mathcal{J}_z, \mathcal{J}_{\overline{z}}] = 0. \tag{3.44}$$

The $(1,0)$-component of the $(s-1)\times(s-1)$-matrix connection of $\mathcal{J}$ is by construction

$$\mathcal{J}_{(z)}{}_{(m)}^{(n)}(z,\overline{z}) \equiv \sum_{R=1}^{s-1} \partial\varpi_{(m)}^{(R)}(z,\overline{z})[\varpi^{-1}]_{(R)}^{(n)}(z,\overline{z}) = \mathcal{R}_{(m+1)}^{(n)}(z,\overline{z}), \qquad m,n = 1,\dots,s-1 \tag{3.45}$$

or more explicitly in matrix form

$$\mathcal{J}_{(z)}(z,\overline{z}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \mathcal{R}_{(s)}^{(1)}(z,\overline{z}) & \mathcal{R}_{(s)}^{(2)}(z,\overline{z}) & \cdots & \cdots & \cdots & \mathcal{R}_{(s)}^{(s-1)}(z,\overline{z}) \end{pmatrix}. \tag{3.46}$$

This matrix turns out to be of Frobenius type (similar to the matrix (3.4)) and therefore $\mathcal{J}_z$ is not a Lie algebra-valued covariant component of an usual connection. Note that the form of the matrix is close to the Drinfeld-Sokolov one [4], but differs by the non-vanishing term $\mathcal{R}_{(s)}^{(s-1)}$. Furthermore, it is useful to notice that

$$\mathcal{R}_{(s)}^{(s-1)}(z,\overline{z}) = \mathrm{Tr}\mathcal{J}_{(z)}(z,\overline{z}) = \mathrm{Tr}(\partial\varpi(z,\overline{z})\varpi^{-1}(z,\overline{z})), \tag{3.47}$$

in accordance with (3.23).

4 B.R.S. approach

In this section, the heuristic presentation of the truncation procedure and its consequences on the formulation of the algebra of Ward operators given in Section 2 is translated into the BRS language. As is well know, this allows to reformulate in more algebraic terms the presentation of a symmetry, and in particular, will give a more universal character of the possible variations on truncated Taylor expansions of the scalar fields Z.

Having still in mind that we are at a fixed given order s for the truncation (2.8) or (3.16), and by recalling Theorem 3.1, we can turn the $s-1$ local parameters $\Gamma^{(\ell)}$ to Faddeev-Popov ($\Phi\Pi$) ghosts $\mathcal{K}^{(l)}$. The variation (2.16) can be recast in a B.R.S. algebraic language as

$$\delta_{\mathcal{W}_s} Z^{(R)}(z,\bar{z}|s) = \sum_{\ell=1}^{s-1} \mathcal{K}^{(\ell)}(z,\bar{z}|s)\partial_{(\ell)} Z^{(R)}(z,\bar{z}|s), \qquad 1 \le R \le s-1, \tag{4.1}$$

where the variation is given by a summand over the independent derivatives up to order $s-1$ due to the DOR of order s. The ghost fields $\mathcal{K}^{(l)}$, of which number is restricted to the range $\ell = 1,\dots,s-1$, serve to define the $\mathcal{W}_s$-algebra relative to the truncation at the level s. We emphasize that the operation $\delta_{\mathcal{W}_s}$ which is required to be nilpotent, depends on the level s of truncation. Accordingly, the ghost parameters $\mathcal{K}^{(l)}$ depend on the truncation process by their number, see (4.1), and generate a $\mathcal{W}$-algebra once the level is fixed. By an argument based on the nilpotency, for $l = 1,\dots,s-1$,

$$\begin{aligned}\delta_{\mathcal{W}_s}\mathcal{K}^{(l)}(z,\bar{z}|s) &=: \sum_{m=1}^{s-1} \mathcal{K}^{(m)}(z,\bar{z}|s)\mathcal{B}^{(l)}_{(m)}(z,\bar{z}|s) \\ &= \sum_{m,n=1}^{s-1} \mathcal{K}^{(m)}(z,\bar{z}|s)\sum_{j=1}^{m}\binom{m}{j}\partial_{(j)}\mathcal{K}^{(n)}(z,\bar{z}|s)\mathcal{R}^{(l)}_{(n+m-j)}(z,\bar{z}|s),\end{aligned} \tag{4.2}$$

where in the r.h.s the dependence in the level s of truncation has been explicitly written. Note also that the products of two underived ghosts drop out by $\Phi\Pi$ charge argument. This variation defines a $(s-1)\times(s-1)$-matrix $\mathcal{B}$ carrying ghost number one, and thus depending on the level s of truncation through the structure functions $\mathcal{R}$ pertaining to that level s. In more details,

$$\mathcal{B}^{(l)}_{(m)}(z,\bar{z}|s) := \sum_{n=1}^{s-1}\sum_{j=0}^{m}\binom{m}{j}\partial_{(m-j)}\mathcal{K}^{(n)}(z,\bar{z}|s)\mathcal{R}^{(l)}_{(n+j)}(z,\bar{z}|s), \tag{4.3}$$

a remarkable combination over the ghosts $\mathcal{K}^{(\ell)}$, $\ell = 1,\dots,s-1$ which could have been readily red of from the variation (2.15). For the Wronskian (3.21) the $\mathcal{W}_s$-algebra extended to

$$\delta_{\mathcal{W}_s}\varpi^{(R)}_{(\ell)}(z,\bar{z}|s) = \sum_{n=1}^{s-1}\mathcal{B}^{(n)}_{(\ell)}(z,\bar{z}|s)\ \varpi^{(R)}_{(n)}(z,\bar{z}|s), \qquad \ell, R = 1,\dots,s-1 \tag{4.4}$$

where the matrix product is understood for $(s-1)\times(s-1)$-matrices. One also have

$$\delta_{\mathcal{W}_s}\varpi^{-1} = -\varpi^{-1}\mathcal{B}, \tag{4.5}$$

and accordingly, using $\delta_{\mathcal{W}}\mathrm{d} + \mathrm{d}\delta_{\mathcal{W}} = 0$,

$$\delta_{\mathcal{W}_s}\mathcal{J} = -\mathrm{d}\mathcal{B} + [\mathcal{B},\mathcal{J}], \tag{4.6}$$

where the bracket is graded on forms with $(s-1)\times(s-1)$-matrix values . The nilpotency property provides first,

$$\delta_{\mathcal{W}_s}\mathcal{B}(z,\bar{z}|s) = \mathcal{B}(z,\bar{z}|s)\mathcal{B}(z,\bar{z}|s) = \mathcal{B}^2(z,\bar{z}|s) \tag{4.7}$$

and second, by ΦΠ argument

$$\delta_{\mathcal{W}_s} \mathrm{Tr}\Big(\mathcal{B}(z,\bar{z}|s)^{(2n+1)}\Big) = 0, \qquad n = 0,1,\dots \tag{4.8}$$

where Tr is the usual trace on matrices. The B.R.S. variation of all the structure functions $\mathcal{R}^{(p)}_{(n)}(z,\bar{z}|s)$ ($p = 1,\dots,s-1$ and even for $n \geq s$) can be directly found from the variation (2.15) to write

$$\delta_{\mathcal{W}_s} \mathcal{R}^{(p)}_{(n)}(z,\bar{z}|s) \;=\; \mathcal{B}^{(p)}_{(n)}(z,\bar{z}|s) - \sum_{q=1}^{s-1} \mathcal{R}^{(q)}_{(n)}(z,\bar{z}|s)\mathcal{B}^{(p)}_{(q)}(z,\bar{z}|s), \tag{4.9}$$

where $\mathcal{B}$ defined above in (4.3) may be extended to a rectangular matrix for lower indices greater than $s-1$, while the upper ones are still lower than this value imposed by the level of truncation, since all the $\mathcal{R}$'s can be gathered into a rectangular matrix. One checks that it is compatible with the case $\mathcal{R}^{(k)}_{(\ell)} = \delta^k_\ell$ which is kept invariant. We stress that while the first $\mathcal{B}$ term of the r.h.s. of the variation (4.9) is a rectangular matrix, the $\mathcal{B}$ under the summand is a squared one. As noted before the matrix $\varpi^R_{(m)}$ can be taken to be a rectangular as well, when $m \geq s$, and the variation (4.4) relative to the algebra $\mathcal{W}_s$ is accordingly modified by

$$\delta_{\mathcal{W}_s} \varpi^{(R)}_{(m)}(z,\bar{z}|s) = \sum_{\ell=1}^{s-1} \mathcal{B}^{(\ell)}_{(m)}(z,\bar{z}|s)\ \varpi^{(R)}_{(\ell)}(z,\bar{z}|s), \qquad R = 1,\dots,s-1 \text{ and } m \geq 1, \tag{4.10}$$

where the matrix $\big(\mathcal{B}^{(\ell)}_{(m)}\big)$ can be rectangular. Supported by the fact that the ghosts $\mathcal{K}^{(\ell)}$ are subordinated to the given level of truncation s, one may now define [28], for $\ell = 1,\dots,s-1$, the ℓ-th derivatives $\partial_{(\ell)} = \left\{\dfrac{\partial}{\partial \mathcal{K}^{(\ell)}}, \delta_{\mathcal{W}_s}\right\}$ as an anticommutator, thus the DOR equation (3.22) is recovered

$$\partial_{(\ell)} \varpi^{(R)}_{(m)}(z,\bar{z}|s) = \sum_{u=1}^{s-1} \mathcal{R}^{(u)}_{(\ell+m)}(z,\bar{z}|s)\varpi^{(R)}_{(u)}(z,\bar{z}|s), \qquad R,\ell = 1,\dots,s-1, \text{ and } m \geq 1 \tag{4.11}$$

This shows that the BRS algebra encapsulates the DOR mechanism just by construction and provides a consistency of the present approach.

Furthermore, from Eq.(4.4) one has

$$\delta_{\mathcal{W}_s} \ln\det \varpi(z,\bar{z}|s) = \mathrm{Tr}\,\mathcal{B}(z,\bar{z}|s) \tag{4.12}$$

which gives for the variation of $\mathcal{R}^{(s-1)}_{(s)}$,

$$\delta_{\mathcal{W}_s} \mathcal{R}^{(s-1)}_{(s)}(z,\bar{z}|s) = \partial\, \mathrm{Tr}\,\mathcal{B}(z,\bar{z}|s) = \mathrm{Tr}\,\partial\,\mathcal{B}(z,\bar{z}|s). \tag{4.13}$$

5 Covariance under holomorphic reparametrization

The covariance property under holomorphic change of local coordinates on the Riemann surface is analyzed for the so far obtained quantities. As it will be shown, this analysis will amount to switching to new ghost fields of a tensorial nature in contrast to that of the $\mathcal{K}$'s.

In view of the patching rules (3.32) under finite holomorphic reparametrizations, it possibly renders more explicit some of the covariance properties of the theory relative to a fixed order s. As it will be shown, the study of covariance will appear as a key step in the construction of $\mathcal{W}$-algebras.

Under finite holomorphic change of charts $z \longrightarrow w(z)$ the covariance property of the $s-1$ scalar fields (emerging from the truncation at level s) writes

$$Z^{(R)}(z,\bar{z}|s) = Z^{(R)}(w,\bar{w}|s), \qquad R = 1,\cdots,s-1 \tag{5.1}$$

and implies that the Wronskian matrix ϖ behaves as a non tensorial covariant quantity

$$\varpi_{(n)}^{(R)}(w,\overline{w}|s) = \sum_{m=1}^{s-1} \Phi_{(n)}^{(m)}(z)\varpi_{(m)}^{(R)}(z,\overline{z}|s), \tag{5.2}$$

and while for its inverse

$$[\varpi^{-1}]_{(R)}^{(n)}(w,\overline{w}|s) = \sum_{m=1}^{s-1} [\varpi^{-1}]_{(R)}^{(m)}(z,\overline{z}|s)\Phi^{-1}{}_{(m)}^{(n)}(z) \tag{5.3}$$

where patching rules are governed by the $(s-1)\times(s-1)$ lower triangular holomorphic matrix $\Phi_{(k)}^{(\ell)}(z)$ for $k,\ell = 1,\dots,s-1$, depending on the Jacobian w' and its derivatives. Its inverse matrix is more easily computable and given by (see (3.32))

$$[\Phi^{-1}]_{(k)}^{(\ell)}(z) = \begin{cases} w^{(k)}(z)\ , & \text{if } \ell = 1 \\ \displaystyle\sum_{r=\ell-1}^{k-1} \frac{(k-1)!\, w^{(k-r)}(z)}{(k-r-1)!} \sum_{\substack{a_1+\cdots+ra_r = r \\ a_1+\cdots+a_r = \ell-1}} \left(\prod_{n=1}^{r} \frac{1}{a_n!} \left(\frac{w^{(n)}(z)}{n!} \right)^{a_n} \right), & k \geq \ell \geq 2 \\ 0, & k < \ell \end{cases} \tag{5.4}$$

with non vanishing determinant, $\det \Phi^{-1}(z) = \big(w'(z)\big)^{s(s-1)/2}$, since the diagonal entries are given by $[\Phi^{-1}]_{(\ell)}^{(\ell)}(z) = (w'(z))^{\ell}$. Note also that the order of the matrix $\Phi_{(k)}^{(\ell)}(z)$ (or that of its inverse as well) is subject to the order s of truncation. Accordingly, since the variation (4.1) relative to the order s has to behave as a scalar, the $s-1$ ghosts of the level s of truncation turn out to be contravariant quantities

$$\mathcal{K}^{(l)}(w,\overline{w}|s) = \mathcal{K}^{(m)}(z,\overline{z}|s)[\Phi^{-1}]_{(m)}^{(l)}(z). \tag{5.5}$$

Taking into account of (5.4), the ghosts $\mathcal{K}^{(m)}(z,\overline{z})$ behave as jets, except for the top one of order $s-1$ which turns out to be a contravariant tensor of order $s-1$. For the sake of completeness, the behaviors of the rectangular matrices $\mathcal{B}_{(p)}^{(l)}(z,\overline{z}|s)$ and $\mathcal{R}_{(p)}^{(l)}(z,\overline{z}|s)$ respectively come from (5.2), (4.4) and (3.16) or (4.9). They are respectively found to be

$$\mathcal{B}(w,\overline{w}|s) = \Phi(z)\mathcal{B}(z,\overline{z}|s)\Phi^{-1}(z), \tag{5.6}$$

and for $\ell \leq s-1$,

$$\mathcal{R}_{(k)}^{(\ell)}(w,\overline{w}|s) = \sum_{m=1}^{k}\sum_{p=\ell}^{s-1} \Phi_{(k)}^{(m)}(z)\mathcal{R}_{(m)}^{(p)}(z,\overline{z}|s)[\Phi^{-1}]_{(p)}^{(\ell)}(z). \tag{5.7}$$

Thanks to the their definition (2.12), one obtains the following identities for $k \leq s-1$,

$$[\Phi^{-1}]_{(k)}^{(k)}(z)\Phi_{(k)}^{(k)}(z) = 1, \quad \text{(no summation)}, \qquad \sum_{u=\ell}^{k} [\Phi^{-1}]_{(k)}^{(u)}(z)\Phi_{(u)}^{(\ell)}(z) = 0, \quad \text{if } \ell < k \leq s-1. \tag{5.8}$$

6 Jets versus tensors, or how to recover $\mathcal{W}$-algebras

The algebra of transformations Eqs.(4.1) and (4.2) are written in terms of ghosts which under holomorphic change of charts (see Eq.(5.5)) behave as jets, thus do not carry any tensorial nature.

We want to show that these transformations encode a structure of $\mathcal{W}_s$-algebra if the D.O.R. mechanism is provided by a truncation at the s-th level. The latter may be implemented by means of a given differential equation (3.1) which serves to generate what it is called in the paper, the Forsyth frames. Since objects of jet nature are heavy to handle, and that (physical) fields are usually considered to be of tensorial nature in some representation space of a symmetry, it is first necessary for the BRS algebra presentation of $\mathcal{W}$-symmetry to switch from the jet-ghosts to tensor ones. This will make some contact with the results on the subject disseminated through the literature [1, 6, 5, 8, 10]. This kind of problem is often encountered in the treatment of a local field theory, and even in the B.R.S.T. quantization scheme, see e.g. [29, 30]. The solution to this problem is obviously not unique since a tensor is defined up to a change of basis among tensors. This is why it must be solved for the moment with the tools at hand.

Let us consider the hypothesis where the $\mathcal{K}$'s are *not* universal, except the top one. Given a level s of truncation, consider the hierarchy of all lower orders of truncation $j+1 \leq s$ which come into the game with their own structure functions. At the level s, if $\vec{Z}$ denotes a vector in $\mathbb{C}P^{s-1}$, one has respectively for the DOR and the variation

$$\partial_{(s)}\vec{Z}(z,\overline{z}|s) = \sum_{\ell=1}^{s-1} \mathcal{R}^{(\ell)}_{(s)}(z,\overline{z}|s)\partial_{(\ell)}\vec{Z}(z,\overline{z}|s), \qquad \delta_{\mathcal{W}_s}\vec{Z}(z,\overline{z}|s) = \sum_{\ell=1}^{s-1} \mathcal{K}^{(\ell)}(z,\overline{z}|s)\partial_{(\ell)}\vec{Z}(z,\overline{z}|s). \quad (6.1)$$

Suppose now that, the DOR is rather implemented at the sub-level $s-1$ on the $s-1$ scalar fields, with

$$\partial_{(s-1)}\vec{Z}(z,\overline{z}|s) = \sum_{\ell=1}^{s-2} \mathcal{R}^{(\ell)}_{(s-1)}(z,\overline{z}|s-1)\partial_{(\ell)}\vec{Z}(z,\overline{z}|s), \quad (6.2)$$

then the above variation becomes can be projected onto that of the level $s-1$

$$\begin{aligned}\delta_{\mathcal{W}_s}\vec{Z}(z,\overline{z}|s) &= \sum_{\ell=1}^{s-2} \Big(\mathcal{K}^{(\ell)}(z,\overline{z}|s) + \mathcal{K}^{(s-1)}(z,\overline{z}|s)\mathcal{R}^{(\ell)}_{(s-1)}(z,\overline{z}|s-1)\Big)\partial_{(\ell)}\vec{Z}(z,\overline{z}|s) \\ &\stackrel{!}{=} \delta_{\mathcal{W}_{s-1}}\vec{Z}(z,\overline{z}|s) = \sum_{\ell=1}^{s-2} \mathcal{K}^{(\ell)}(z,\overline{z}|s-1)\partial_{(\ell)}\vec{Z}(z,\overline{z}|s). \end{aligned} \quad (6.3)$$

Upon requiring that $\delta_{\mathcal{W}_s}\vec{Z}(z,\overline{z}|s) = \delta_{\mathcal{W}_{s-1}}\vec{Z}(z,\overline{z}|s)$, one can identify the top tensorial ghost of the level $s-1$ in terms of those of the upper level s through the structure functions of the level $s-1$, by

$$\mathcal{K}^{(s-2)}(z,\overline{z}|s-1) = \mathcal{K}^{(s-2)}(z,\overline{z}|s) + \mathcal{K}^{(s-1)}(z,\overline{z}|s)\mathcal{R}^{(s-2)}_{(s-1)}(z,\overline{z}|s-1). \quad (6.4)$$

Repeating the DOR from the level s to an arbitrary sub-level j, with $2 \leq j \leq s-1$, the requirement that $\delta_{\mathcal{W}_s}\vec{Z}(z,\overline{z}|s) = \delta_{\mathcal{W}_j}\vec{Z}(z,\overline{z}|s)$ yields for each top tensorial ghost

$$\begin{aligned}\mathcal{K}^{(j-1)}(z,\overline{z}|j) &= \mathcal{K}^{(j-1)}(z,\overline{z}|s) + \sum_{m=j}^{s-1} \mathcal{K}^{(m)}(z,\overline{z}|s)\mathcal{R}^{(j-1)}_{(m)}(z,\overline{z}|j) \\ &= \sum_{m=j-1}^{s-1} \mathcal{K}^{(m)}(z,\overline{z}|s)\mathcal{R}^{(j-1)}_{(m)}(z,\overline{z}|j). \end{aligned} \quad (6.5)$$

Next, taking the the ghost of highest conformal weight in each of the sub-algebras, one generates a hierarchy of j-contravariant conformal tensors as

$$\mathcal{C}^{(j)}(z,\overline{z}) := \mathcal{K}^{(j)}(z,\overline{z}|j+1)\ . \quad (6.6)$$

All the above considerations suggest to take as an ansatz for the j-contravariant ghost conformal tensors the following pretty tricky linear combination

$$\mathcal{C}^{(j)}(z,\overline{z}) = \sum_{m=j}^{s-1} \mathcal{K}^{(m)}(z,\overline{z}|s)\mathcal{R}^{(j)}_{(m)}(z,\overline{z}|j+1), \qquad \text{for } 1 \leq j \leq s-1, \quad (6.7)$$

where the lower orders of truncation (implemented by the $\mathrm{DOR}_{s-1\to j}$) crucially enter the construction. Due to the requirement that $\delta_{\mathcal{W}_s}\vec{Z}(z,\overline{z}|s) = \delta_{\mathcal{W}_k}\vec{Z}(z,\overline{z}|s)$, for $k = 2,\ldots,s$, it is worthwhile to emphasize that the tensorial ghosts $\mathcal{C}^{(j)}$ carry an universal nature (regarding all the hierarchy of $\mathcal{W}_j$-algebras), in the sense that for $j = 1, \ldots, k-1$, $k \geq j+1$ one has the identity

$$\mathcal{C}^{(j)}(z,\bar{z}) = \sum_{m=j}^{s-1} \mathcal{K}^{(m)}(z,\bar{z}|s)\mathcal{R}^{(j)}_{(m)}(z,\bar{z}|j+1) = \sum_{m=j}^{k-1} \mathcal{K}^{(m)}(z,\bar{z}|k)\mathcal{R}^{(j)}_{(m)}(z,\bar{z}|j+1). \tag{6.8}$$

The latter (which generalizes (6.7)) shows that the $\mathcal{C}^{(j)}$'s do not depend on the level k of truncation for $k \geq j$. This strongly suggests that the tensorial ghosts $\mathcal{C}^{(j)}$'s are of universal nature. Moreover, one gets for the $\mathrm{DOR}_{s-1\to k-1}$ and with $\ell = 1, \ldots, k-1$

$$\mathcal{K}^{(\ell)}(z,\bar{z}|k) = \mathcal{K}^{(\ell)}(z,\bar{z}|s) + \sum_{m=k}^{s-1} \mathcal{K}^{(m)}(z,\bar{z}|s)\mathcal{R}^{(\ell)}_{(m)}(z,\bar{z}|k) = \sum_{m=1}^{s-1} \mathcal{K}^{(m)}(z,\bar{z}|s)\mathcal{R}^{(\ell)}_{(m)}(z,\bar{z}|k). \tag{6.9}$$

Next, by comparing in order to guarantee the transitivity property, $\mathrm{DOR}_{s-1\to j}$ with $\mathrm{DOR}_{s-1\to k-1}$ followed by $\mathrm{DOR}_{k-1\to j}$, the structure functions must verify for any $m \geq k$ with $j+1 \leq k \leq s-1$

$$\mathcal{R}^{(j)}_{(m)}(z,\bar{z}|j+1) = \sum_{\ell=j}^{k-1} \mathcal{R}^{(\ell)}_{(m)}(z,\bar{z}|k)\mathcal{R}^{(j)}_{(\ell)}(z,\bar{z}|j+1), \tag{6.10}$$

an identity which has to be checked with the help of (3.34) and both the choices of j and of $k-1$ coordinates among the $s-1$ coordinates given by $\vec{Z} \in \mathbb{C}P^{s-1}$ (identity between determinants). These two choices of sub-coordinates define sub-manifolds in $\mathbb{C}P^{s-1}$. This phenomenon is the signature of the presence of flag manifolds denoted by $F_{j\cdots s-2}\mathbb{C}^{s-1}$ over $\mathbb{C}P^{s-1}$, a geometric concept already mentioned as related to $\mathcal{W}$-algebras in [5]. In particular the choice $j = 1$ and $k = s$ in (6.10) corresponds to the whole hierarchy (6.7) of the $\mathcal{C}$ ghosts and is associated to the flag manifold $F_{12\cdots s-2}\mathbb{C}^{s-1}$.

Owing to the above considerations, let us define now, for a fixed level s, the following nilpotent operator $\delta_{\mathcal{W}_s} = \bigoplus_{\ell=2}^{s-1} \delta_{\mathcal{W}_\ell}$, with $\delta^2_{\mathcal{W}_\ell} = 0$ and $\{\delta_{\mathcal{W}_k}, \delta_{\mathcal{W}_\ell}\} = 0$, which is in some sense filtrated by the various sub-DOR's relative to the flag sub-manifold $F_{12\cdots s-2}\mathbb{C}^{s-1}$. Then the task is to figure out the variations $\delta_{\mathcal{W}_s}\mathcal{C}^{(j)}$ for $j = 1, \ldots, s-1$ in terms of the tensorial $\mathcal{C}$'s themselves.

Note that the tensorial ansatz (6.7) gives an universal character to each of the tensorial top ghosts $\mathcal{C}^{(\ell-1)}(z,\bar{z}) := \mathcal{K}^{(\ell-1)}(z,\bar{z}|\ell)$ of each sub-levels. By virtue of (6.9), the latter linearly depend on both the jet ghosts $\mathcal{K}(z,\bar{z}|\ell)$'s of the top order s of truncation and the structure functions relative to the various truncations up to order $s-1$.

In the course of the checking that the $\mathcal{C}^{(j)}$'s are indeed j-contravariant conformal tensors, (5.5), (5.7) and the identities (5.8) were repeatedly used. The tensor character of $\mathcal{C}^{(j)}$ is secured by the choice of $\mathcal{R}^{(j)}_{(n)}(z,\bar{z}|j+1)$ with maximum upper index j relatively to the truncation of level $j+1$. This is possible if one picks up these objects from the whole underlying D.O.R. decompositions with a truncation mechanism at each level lower than s. So the price to pay is the introduction of all the $\mathcal{R}^{(j)}_{(n)}(z,\bar{z}|j+1)$ coefficients relative to all the (sub-)truncations from $j = 1$ to $j = s-1$. The latter could have been implemented by a hierarchy of differential equations of the type (3.1). The ansatz (6.7) is a linear system in a Gauss form which is easily inverted as

$$\begin{aligned} \mathcal{K}^{(\ell)}(z,\bar{z}|s) &= \sum_{m=\ell}^{s-1} \mathcal{C}^{(m)}(z,\bar{z})\,\mathcal{U}^{(\ell)}_{(m)}(z,\bar{z}|\ell+1,\ldots,s-1), \qquad \ell = 1,\ldots,s-2 \\ \mathcal{K}^{(s-1)}(z,\bar{z}|s) &= \mathcal{C}^{(s-1)}(z,\bar{z}), \end{aligned} \tag{6.11}$$

where $\mathcal{U}^{(\ell)}_{(m)}(z,\bar{z}|\ell+1,\ldots,s-1)$ is the coefficient of the inverse upper triangular matrix which depends polynomially on structure functions pertaining to the sub-levels from $\ell+1$ to $s-1$. More explicitly, for

$k,\ell=1,\ldots,s-1$

$$\mathcal{U}^{(\ell)}_{(k)}(z,\overline{z}|\ell+1,\ldots,s-1)=\begin{cases}1 & \text{if } k=\ell\\ 0 & \text{if } k<\ell\\ \displaystyle\sum_{j=1}^{k-\ell}(-1)^j\Big[\prod_{i=1}^{j}\mathcal{R}^{(\ell_i)}_{(k_i)}(z,\overline{z}|\ell_i+1)\Big]\Bigg|_{\substack{\ell_1=\ell,\ k_j=k\\ k_i>\ell_i\\ k_i=\ell_{i+1}\ \text{for}\ k-\ell\geq 2}} & \text{if } k>\ell\end{cases}\tag{6.12}$$

The variation of the $\mathcal{C}^{(j)}$'s given by (6.7) is computed upon using the variation (4.2) for the level s and the variation (4.9) for all the sub-levels up to $s-1$ owing to the DOR filtrations. It writes

$$\delta_{\mathcal{W}_s}\mathcal{C}^{(j)}(z,\overline{z})=\sum_{\ell=j}^{s-1}\Big(\delta_{\mathcal{W}_s}\mathcal{K}^{(\ell)}(z,\overline{z}|s)\mathcal{R}^{(j)}_{(\ell)}(z,\overline{z}|j+1)-\mathcal{K}^{(\ell)}(z,\overline{z}|s)\delta_{\mathcal{W}_{j+1}}\mathcal{R}^{(j)}_{(\ell)}(z,\overline{z}|j+1)\Big).\tag{6.13}$$

and after some algebra we get the following variation,

$$\begin{aligned}\delta_{\mathcal{W}_s}\mathcal{C}^{(j)}(z,\overline{z})\ =&\ \sum_{n=1}^{s-1}\sum_{a=n}^{s-1}\sum_{b=1}^{s-1}\sum_{r=0}^{n}\mathcal{C}^{(a)}(z,\overline{z})\partial_{(r)}\mathcal{C}^{(b)}(z,\overline{z})\,\mathcal{U}^{(n)}_{(a)}(z,\overline{z}|a+1,\ldots,s-1)\sum_{\ell=1}^{b}\sum_{k=r}^{n}\binom{k}{r}\binom{n}{k}\\ &\times\partial_{(k-r)}\mathcal{U}^{(\ell)}_{(b)}(z,\overline{z}|b+1,\ldots,s-1)\left(\sum_{p=j}^{s-1}\mathcal{R}^{(p)}_{(n+\ell-k)}(z,\overline{z}|s)\mathcal{R}^{(j)}_{(p)}(z,\overline{z}|j+1)\right)\\ &+\sum_{n=j}^{s-1}\sum_{a=n}^{s-1}\sum_{b=1}^{j}\Bigg[\sum_{u=1}^{j}\sum_{r=0}^{u}\mathcal{C}^{(a)}(z,\overline{z})\partial_{(r)}\mathcal{C}^{(b)}(z,\overline{z})\,\mathcal{U}^{(n)}_{(a)}(z,\overline{z}|a+1,\ldots,s-1)\sum_{\ell=1}^{b}\sum_{k=r}^{u}\binom{k}{r}\binom{u}{k}\\ &\times\partial_{(k-r)}\mathcal{U}^{(\ell)}_{(b)}(z,\overline{z}|b+1,\ldots,j)\mathcal{R}^{(u)}_{(n)}(z,\overline{z}|j+1)\mathcal{R}^{(j)}_{(u+\ell-k)}(z,\overline{z}|j+1)\\ &-\sum_{r=0}^{n}\mathcal{C}^{(a)}(z,\overline{z})\partial_{(r)}\mathcal{C}^{(b)}(z,\overline{z})\,\mathcal{U}^{(n)}_{(a)}(z,\overline{z}|a+1,\ldots,s-1)\sum_{\ell=1}^{b}\sum_{k=r}^{n}\binom{k}{r}\\ &\times\partial_{(k-r)}\mathcal{U}^{(\ell)}_{(b)}(z,\overline{z}|b+1,\ldots,j)\mathcal{R}^{(j)}_{(n+\ell-k)}(z,\overline{z}|j+1)\Bigg].\end{aligned}\tag{6.14}$$

Equation (6.14) can be disassembled into:

$$\delta_{\mathcal{W}_s}\mathcal{C}^{(j)}(z,\overline{z})\ \equiv\ \sum_{n=1}^{j}n\,\mathcal{C}^{(n)}(z,\overline{z})\partial\mathcal{C}^{(j-n+1)}(z,\overline{z})+\mathcal{X}^{(j)}(z,\overline{z}|j+1,\ldots,s),\tag{6.15}$$

where the first summand looks like the variation coming from a symplectic approach [11] to $\mathcal{W}$-algebra.

It is clear that the last term $\mathcal{X}$ is related to the whole symmetry in the sense that it is a tensorial differential expression of ghost grading two in the various structure functions $\mathcal{R}$'s of the sub-levels. Due to the nilpotency of the $\delta_{\mathcal{W}_s}$ BRS operation, the $\mathcal{X}^{(j)}$'s defined in (6.15) do transform according to

$$\begin{aligned}\delta_{\mathcal{W}_s}\mathcal{X}^{(j)}(z,\overline{z}|j+1,\ldots,s)\ =&\ \sum_{n=1}^{j}n\Big(\mathcal{C}^{(n)}(z,\overline{z})\partial\mathcal{X}^{(j-n+1)}(z,\overline{z}|j+1,\ldots,s)\\ &-\mathcal{X}^{(n)}(z,\overline{z}|j+1,\ldots,s)\partial\mathcal{C}^{(j-n+1)}(z,\overline{z})\Big).\end{aligned}\tag{6.16}$$

The full completion of the last equation (6.16) amounts to introducing (together with their $\mathcal{W}_s$-variations) all a set of primary fields ($\mathcal{W}$-currents) which belong to the tower of all the nested sub-algebras according to the DOR filtration and the respective variations for each sub-levels. This provides a general solution for any j, and is, according to our opinion, the most general explicit expression given, up to now, for *any* $\mathcal{W}$-algebra in a B.R.S setting. Of course, the generic expression (6.14) contains the $\mathcal{R}$ reduction coefficients of *all* the Forsyth sub-frames. However, the variations given by Eq.(6.14) do not generally coincide with the ones found in the literature: this fact will be illustrated in the next two examples. In order to recover the familiar expressions [12, 23] nontrivial redefinitions of the tensorial ghosts involving derivative terms must be performed. The ansatz (6.11) relating the jet ghosts to the tensorial ones is recast into the form [15]:

$$\text{for } \ell = 1, \ldots, s-2, \qquad \mathcal{K}^{(\ell)}(z,\bar{z}|s) = \mathcal{C}^{(\ell)}(z,\bar{z}) + \sum_{p=\ell+1}^{s-1} \sum_{r=0}^{p-\ell} \partial_{(r)} \mathcal{C}^{(p)}(z,\bar{z}) \mathcal{T}^{(r,\ell)}_{(p)}(z,\bar{z}|s), \tag{6.17}$$

where derivatives in the tensorial ghosts explicitly enter and where $\mathcal{T}^{(b,m)}_{(p)}(z,\bar{z}|s)$ depends only on the structure functions $\mathcal{R}^{(\ell)}_{(s)}(z,\bar{z}|s)$, $\ell = 1, \ldots, s-1$ with $\mathcal{T}^{(r,m)}_{(p)}(z,\bar{z}|s) = 0$ for $p < m$ and $\mathcal{T}^{(r,m)}_{(p)}(z,\bar{z}|s) = \delta^{(r)}_{(0)}$ for $p = m$.

In this case, the new $\mathcal{X}^{(j)}$'s will depend only on the structure functions of the level s. So, we stress, that this is a consequence of the technical difficulties coming from the jets to tensor reduction, and it is not a problem of the $\mathcal{W}$-symmetry itself.

In fact this is deduced by searching the algebraic conditions (in terms of ghosts and their derivatives, considered as independent fields), which put to zero, in all the equations (6.14) all the terms containing the structure functions of the sub-levels $\mathcal{R}(z,\bar{z}|j)$, for $j < s$. For s of reasonable order, we find that the number of vanishing conditions allows to get an unique solution, and numerically provides the $\mathcal{W}$ examples found in the literature.

The present upshot greatly improves some previous work [27, 31] in the sense that it is now possible to construct explicitly the $\mathcal{X}$-term which breaks by truncation the $\mathcal{W}_\infty$-symmetry governed by an underlying symplectomorphism symmetry [11, 32] to a finite $\mathcal{W}_s$-algebra.

To distinguish in a clear way a realization of such a $\mathcal{W}_s$-structure, the $\mathcal{W}_3$ and $\mathcal{W}_4$ cases will be next computed in great details. Then, despite the lack of well settled examples in the literature for (even if some examples exist [33, 34]) $\mathcal{W}_s$ ($s \geq 5$), remarkable results in [10] will allow to find out a general setting.

6.1 The $\mathcal{W}_3$ example

According to the general construction, one has with $s = 3$ as top level, the two tensorial ghosts

$$\begin{aligned} \mathcal{C}^{(1)}(z,\bar{z}) &= \mathcal{K}^{(1)}(z,\bar{z}|2) = \mathcal{K}^{(1)}(z,\bar{z}|3) + \mathcal{K}^{(2)}(z,\bar{z}|3)\mathcal{R}^{(1)}_{(2)}(z,\bar{z}|2) \\ \mathcal{C}^{(2)}(z,\bar{z}) &= \mathcal{K}^{(2)}(z,\bar{z}|3) \end{aligned} \tag{6.18}$$

and their holomorphically covariant variations according to the DOR filtration read

$$\begin{aligned} \delta_{\mathcal{W}_3} \mathcal{C}^{(2)}(z,\bar{z}) &= \mathcal{C}^{(1)}(z,\bar{z})\partial\mathcal{C}^{(2)}(z,\bar{z}) + 2\mathcal{C}^{(2)}(z,\bar{z})\partial\mathcal{C}^{(1)}(z,\bar{z}) + \mathcal{C}^{(2)}(z,\bar{z})\partial_{(2)}\mathcal{C}^{(2)}(z,\bar{z}) \\ &\quad + \mathcal{C}^{(2)}(z,\bar{z})\partial\mathcal{C}^{(2)}(z,\bar{z})\Big[2\mathcal{R}^{(2)}_{(3)}(z,\bar{z}|3) - 3\mathcal{R}^{(1)}_{(2)}(z,\bar{z}|2)\Big] \end{aligned} \tag{6.19}$$

$$\begin{aligned} \delta_{\mathcal{W}_3} \mathcal{C}^{(1)}(z,\bar{z}) &= \mathcal{C}^{(1)}(z,\bar{z})\partial\mathcal{C}^{(1)}(z,\bar{z}) + 2\mathcal{C}^{(2)}(z,\bar{z})\partial\mathcal{C}^{(2)}(z,\bar{z})\Big[\mathcal{R}^{(1)}_{(3)}(z,\bar{z}|3) - \partial\mathcal{R}^{(1)}_{(2)}(z,\bar{z}|2) \\ &\quad + \mathcal{R}^{(2)}_{(3)}(z,\bar{z}|3)\mathcal{R}^{(1)}_{(2)}(z,\bar{z}|2) - \big(\mathcal{R}^{(1)}_{(2)}(z,\bar{z}|2)\big)^2\Big], \end{aligned} \tag{6.20}$$

where in the course of the calculation eq.(3.31) has been used. Performing the holomorphically covariant change of generators

$$\mathcal{C}^{(1)}(z,\bar{z}) \to \widetilde{\mathcal{C}}^{(1)}(z,\bar{z}) := \mathcal{C}^{(1)}(z,\bar{z}) + \tfrac{1}{2}\partial\mathcal{C}^{(2)}(z,\bar{z}) + \mathcal{C}^{(2)}(z,\bar{z})\Big[\tfrac{2}{3}\mathcal{R}^{(2)}_{(3)}(z,\bar{z}|3) - \mathcal{R}^{(1)}_{(2)}(z,\bar{z}|2)\Big], \tag{6.21}$$

allows to removing the $\mathcal{R}^{(1)}_{(2)}(z,\overline{z}|2)$ dependence and (6.11) rewrites

$$\begin{aligned}\mathcal{K}^{(1)}(z,\overline{z}|3) &= \widetilde{\mathcal{C}}^{(1)}(z,\overline{z}) - \tfrac{1}{2}\partial\mathcal{C}^{(2)}(z,\overline{z}) - \tfrac{2}{3}\mathcal{C}^{(2)}(z,\overline{z})\mathcal{R}^{(2)}_{(3)}(z,\overline{z}|3)\\ \mathcal{K}^{(2)}(z,\overline{z}|3) &= \mathcal{C}^{(2)}(z,\overline{z}),\end{aligned}\tag{6.22}$$

and depends only on the level $s=3$. Next, we get the well known transformations [12, 27] upon redefining $\widetilde{\mathcal{C}}^{(2)} = \frac{1}{2}\,\mathcal{C}^{(2)}$ by a numerical rescaling

$$\begin{aligned}\delta_{\mathcal{W}_3}\widetilde{\mathcal{C}}^{(1)}(z,\overline{z}) &= \widetilde{\mathcal{C}}^{(1)}(z,\overline{z})\partial\widetilde{\mathcal{C}}^{(1)}(z,\overline{z}) - \tfrac{4}{3}\mathcal{T}_{(2)}(z,\overline{z}|3)\,\mathcal{C}^{(2)}(z,\overline{z})\partial\mathcal{C}^{(2)}(z,\overline{z})\\ &\quad + \tfrac{1}{4}\Big(\partial\mathcal{C}^{(2)}(z,\overline{z})\partial^2\mathcal{C}^{(2)}(z,\overline{z}) - \tfrac{2}{3}\mathcal{C}^{(2)}(z,\overline{z})\partial^3\mathcal{C}^{(2)}(z,\overline{z})\Big)\\ \delta_{\mathcal{W}_3}\mathcal{C}^{(2)}(z,\overline{z}) &= \widetilde{\mathcal{C}}^{(1)}(z,\overline{z})\partial\mathcal{C}^{(2)}(z,\overline{z}) + 2\mathcal{C}^{(2)}(z,\overline{z})\partial\widetilde{\mathcal{C}}^{(1)}(z,\overline{z})\end{aligned}\tag{6.23}$$

where the expected combination (3.37) which introduces into the game a projective connection, is recovered for the case $s=3$,

$$\mathcal{T}_{(2)}(z,\overline{z}|3) = \tfrac{1}{2}\Big(\partial\mathcal{R}^{(2)}_{(3)}(z,\overline{z}|3) - \tfrac{1}{3}\big(\mathcal{R}^{(2)}_{(3)}(z,\overline{z}|3)\big)^2 - \mathcal{R}^{(1)}_{(3)}(z,\overline{z}|3)\Big) = \tfrac{1}{2}a^{(3)}_{(2)}(z,\overline{z}),\tag{6.24}$$

an expression which depends on the level $s=3$ only. The remarkable fact in the course of the computation of (6.23) is that all the quantities pertaining to the sub-order 2 of truncation have disappeared thanks to the change of generators (6.21) in the tensorial sector.

Comforted into this approach, one can compute also the variation of $\mathcal{T}_{(2)}(z,\overline{z}|3)$. After a straightforward but rather tedious calculation one successively obtains

$$\begin{aligned}\delta_{\mathcal{W}_3}\mathcal{T}_{(2)}(z,\overline{z}|3) &= \Big(\partial_{(3)}\mathcal{K}^{(1)} + 2\,\mathcal{T}_{(2)}\partial\mathcal{K}^{(1)} + \mathcal{K}^{(1)}\mathcal{T}_{(2)}\Big)(z,\overline{z}|3)\\ &+ \Big(\mathcal{K}^{(2)}\Big[\partial_{(3)}\mathcal{R}^{(2)}_{(3)} + \partial_{(2)}\mathcal{R}^{(1)}_{(3)} + \partial_{(2)}\big(\mathcal{R}^{(2)}_{(3)}\big)^2 - \tfrac{2}{3}\mathcal{R}^{(2)}_{(3)}\partial_{(2)}\mathcal{R}^{(2)}_{(3)} - \tfrac{2}{3}\mathcal{R}^{(2)}_{(3)}\partial\big(\mathcal{R}^{(2)}_{(3)}\big)^2 - 2\mathcal{R}^{(1)}_{(3)}\partial\mathcal{R}^{(2)}_{(3)} - \tfrac{4}{3}\mathcal{R}^{(2)}_{(3)}\partial\mathcal{R}^{(1)}_{(3)}\Big]\\ &\quad + \partial\mathcal{K}^{(2)}\Big[4\partial_{(2)}\mathcal{R}^{(2)}_{(3)} + \partial\mathcal{R}^{(1)}_{(3)} + \partial\big(\mathcal{R}^{(2)}_{(3)}\big)^2 - \tfrac{2}{3}\big(\mathcal{R}^{(2)}_{(3)}\big)^3 - \tfrac{7}{3}\mathcal{R}^{(2)}_{(3)}\mathcal{R}^{(1)}_{(3)}\Big]\\ &\quad + \partial_{(2)}\mathcal{K}^{(2)}\Big[5\partial\mathcal{R}^{(2)}_{(3)} - \mathcal{R}^{(1)}_{(3)} - \tfrac{1}{3}\big(\mathcal{R}^{(2)}_{(3)}\big)^2\Big] + \tfrac{4}{3}\mathcal{R}^{(2)}_{(3)}\partial_{(3)}\mathcal{K}^{(2)} + \partial_{(4)}\mathcal{K}^{(2)}\Big)(z,\overline{z}|3),\end{aligned}\tag{6.25}$$

in terms of the two $\mathcal{K}$ ghosts and the structure functions relative to the level $s=3$. According to (6.22), this variation can be re-expressed in terms of the two tensorial $\widetilde{\mathcal{C}}$ ghosts as

$$\begin{aligned}\delta_{\mathcal{W}_3}\mathcal{T}_{(2)}(z,\overline{z}|3) &= \partial_{(3)}\widetilde{\mathcal{C}}^{(1)}(z,\overline{z}) + 2\,\mathcal{T}_{(2)}(z,\overline{z}|3)\partial\widetilde{\mathcal{C}}^{(1)}(z,\overline{z}) + \widetilde{\mathcal{C}}^{(1)}(z,\overline{z})\mathcal{T}_{(2)}(z,\overline{z}|3)\\ &\quad - 2\,\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\partial\Big[\tfrac{1}{6}\partial_{(2)}\mathcal{R}^{(2)}_{(3)} - \tfrac{1}{6}\partial\big(\mathcal{R}^{(2)}_{(3)}\big)^2 + \tfrac{1}{3}\mathcal{R}^{(2)}_{(3)}\mathcal{R}^{(1)}_{(3)} + \tfrac{2}{27}\big(\mathcal{R}^{(2)}_{(3)}\big)^3 - \tfrac{1}{2}\partial\mathcal{R}^{(1)}_{(3)}\Big](z,\overline{z}|3)\\ &\quad - 3\,\partial\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\Big[\tfrac{1}{6}\partial_{(2)}\mathcal{R}^{(2)}_{(3)} - \tfrac{1}{6}\partial\big(\mathcal{R}^{(2)}_{(3)}\big)^2 + \tfrac{1}{3}\mathcal{R}^{(2)}_{(3)}\mathcal{R}^{(1)}_{(3)} + \tfrac{2}{27}\big(\mathcal{R}^{(2)}_{(3)}\big)^3 - \tfrac{1}{2}\partial\mathcal{R}^{(1)}_{(3)}\Big](z,\overline{z}|3).\end{aligned}\tag{6.26}$$

The expression between the brackets corresponds to the associated W_3-current as a cubic differential relative to the level $s=3$, (up to a factor)

$$\begin{aligned}8\,W_{(3)}(z,\overline{z}|3) &= \Big(\tfrac{1}{6}\partial_{(2)}\mathcal{R}^{(2)}_{(3)} - \tfrac{1}{6}\partial\big(\mathcal{R}^{(2)}_{(3)}\big)^2 + \tfrac{1}{3}\mathcal{R}^{(2)}_{(3)}\mathcal{R}^{(1)}_{(3)} + \tfrac{2}{27}\big(\mathcal{R}^{(2)}_{(3)}\big)^3 - \tfrac{1}{2}\partial\mathcal{R}^{(1)}_{(3)}\Big)(z,\overline{z}|3)\\ &= \Big(\tfrac{1}{2}\partial a^{(3)}_{(2)} - a^{(3)}_{(3)}\Big)(z,\overline{z}),\end{aligned}\tag{6.27}$$

where the $a^{(3)}$'s were given in (3.39).

To sump up, the general conformally covariant differential operator (3.1) for $s=3$ can be recast in terms of the two $\mathcal{W}$-currents

$$L_3(z,\overline{z}) = \partial_{(3)} + 2\mathcal{T}_{(2)}(z,\overline{z}|3)\partial + \partial\mathcal{T}_{(2)}(z,\overline{z}|3) - 8\,W_{(3)}(z,\overline{z}|3)\tag{6.28}$$

where the last type $(3,0)$-term indicates the difference with the so-called Bol operator of order 3, see e.g. [24].

At the level of the BRS differential algebra, dropping out the $\widetilde{}$ for the tensorial ghosts, one has an explicit realization of the so-called principal $\mathcal{W}_3$-algebra, related to what it is called the pure $\mathcal{W}_3$-gravity [35]. The nilpotent BRS algebra for $\mathcal{W}_3$ writes in terms of a $\mathcal{S}$-operation acting on $\mathcal{T}$ and $W_{(3)}$ which are of spin 2 and spin 3 $\mathcal{W}$-currents, respectively and of two conformal ghost fields $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}$

$$
\begin{aligned}
\mathcal{S}\mathcal{C}^{(1)} &= \mathcal{C}^{(1)}\partial\mathcal{C}^{(1)} + \partial\mathcal{C}^{(2)}\partial^2\mathcal{C}^{(2)} - \tfrac{2}{3}\mathcal{C}^{(2)}\partial^3\mathcal{C}^{(2)} - \tfrac{16}{3}\mathcal{T}\mathcal{C}^{(2)}\partial\mathcal{C}^{(2)} \\
\mathcal{S}\mathcal{C}^{(2)} &= \mathcal{C}^{(1)}\partial\mathcal{C}^{(2)} + 2\mathcal{C}^{(2)}\partial\mathcal{C}^{(1)} \\
\mathcal{S}\mathcal{T} &= \partial^3\mathcal{C}^{(1)} + 2\mathcal{T}\partial\mathcal{C}^{(1)} + \mathcal{C}^{(1)}\partial\mathcal{T} - 8(2\mathcal{C}^{(2)}\partial W_{(3)} + 3\,W_{(3)}\partial\mathcal{C}^{(2)}) \\
\mathcal{S}W_{(3)} &= \tfrac{1}{24}\Big(\partial^5\mathcal{C}^{(2)} + 2\mathcal{C}^{(2)}\partial^3\mathcal{T} + 10\mathcal{T}\partial^3\mathcal{C}^{(2)} + 15\partial\mathcal{T}\partial^2\mathcal{C}^{(2)} + 9\partial^2\mathcal{T}\partial\mathcal{C}^{(2)} \\
&\qquad + 16\mathcal{T}\partial\mathcal{T}\mathcal{C}^{(2)} + 16\mathcal{T}^2\partial\mathcal{C}^{(2)}\Big) + \mathcal{C}^{(1)}\partial W_{(3)} + 3\,W_{(3)}\partial\mathcal{C}^{(1)}\ .
\end{aligned}
\tag{6.29}
$$

However, at the practical level, the $\mathcal{W}_3$ case stands as a particular example in the sense that the change of generators emerges by itself. But for the instance of $\mathcal{W}_4$, it is not evident at first sight, to figure out which change of generators for $C^{(1)}$ and $C^{(2)}$ must be performed. For the moment, there is no a general criterion at our disposal giving any guidance on that step. Nevertheless, an explicit realization of the so-called principal $\mathcal{W}_4$-algebra can be constructed along both the ideas of respecting the covariance and the dependence of the top level $s = 4$ only. These two main ideas are the crux of all the construction and must be explained in a more geometric setup.

6.2 The $\mathcal{W}_4$ case

According to the general construction, this time one has with $s = 4$ as top level the three tensorial ghosts

$$
\begin{aligned}
\mathcal{C}^{(1)}(z,\bar z) &= \mathcal{K}^{(1)}(z,\bar z|2) = \mathcal{K}^{(1)}(z,\bar z|4) + \mathcal{K}^{(2)}(z,\bar z|4)\mathcal{R}^{(1)}_{(2)}(z,\bar z|2) + \mathcal{K}^{(3)}(z,\bar z|4)\mathcal{R}^{(1)}_{(3)}(z,\bar z|2) \\
\mathcal{C}^{(2)}(z,\bar z) &= \mathcal{K}^{(2)}(z,\bar z|3) = \mathcal{K}^{(2)}(z,\bar z|4) + \mathcal{K}^{(3)}(z,\bar z|4)\mathcal{R}^{(2)}_{(3)}(z,\bar z|3) \\
\mathcal{C}^{(3)}(z,\bar z) &= \mathcal{K}^{(3)}(z,\bar z|4).
\end{aligned}
\tag{6.30}
$$

The holomorphically covariant variation according to the DOR filtration of the top ghost is found to be

$$
\begin{aligned}
\delta_{\mathcal{W}_4}\mathcal{C}^{(3)}(z,\bar z) &= \big(\mathcal{C}^{(1)}\partial\mathcal{C}^{(3)} + 3\mathcal{C}^{(3)}\partial\mathcal{C}^{(1)} + 2\mathcal{C}^{(2)}\partial\mathcal{C}^{(2)} + \mathcal{C}^{(3)}\partial_{(3)}\mathcal{C}^{(3)} + 3\mathcal{C}^{(3)}\partial_{(2)}\mathcal{C}^{(2)} + \mathcal{C}^{(2)}\partial_{(2)}\mathcal{C}^{(3)}\big)(z,\bar z) \\
&+ (\mathcal{C}^{(3)}\partial\mathcal{C}^{(3)})(z,\bar z)\Big[3\partial\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) + 3\mathcal{R}^{(2)}_{(4)}(z,\bar z|4) + 3\big(\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\big)^2 - 6\partial\mathcal{R}^{(2)}_{(3)}(z,\bar z|3) \\
&\qquad - 4\partial\mathcal{R}^{(1)}_{(2)}(z,\bar z|2) - 5\mathcal{R}^{(2)}_{(3)}(z,\bar z|3)\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - 4\big(\mathcal{R}^{(1)}_{(2)}(z,\bar z|2)\big)^2 \\
&\qquad + 4\mathcal{R}^{(1)}_{(2)}(z,\bar z|2)\mathcal{R}^{(2)}_{(3)}(z,\bar z|3) + 2\big(\mathcal{R}^{(2)}_{(3)}(z,\bar z|3)\big)^2\Big] \\
&+ (\mathcal{C}^{(3)}\partial_{(2)}\mathcal{C}^{(3)})(z,\bar z)\Big[3\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - 4\mathcal{R}^{(2)}_{(3)}(z,\bar z|3)\Big] \\
&+ (\mathcal{C}^{(3)}\mathcal{C}^{(2)})(z,\bar z)\Big[2\partial\mathcal{R}^{(2)}_{(3)}(z,\bar z|3) - 3\partial\mathcal{R}^{(1)}_{(2)}(z,\bar z|2)\Big] \\
&+ (\mathcal{C}^{(2)}\partial\mathcal{C}^{(3)})(z,\bar z)\Big[2\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - 2\mathcal{R}^{(2)}_{(3)}(z,\bar z|3) - \mathcal{R}^{(1)}_{(2)}(z,\bar z|2)\Big] \\
&+ (\mathcal{C}^{(3)}\partial\mathcal{C}^{(2)})(z,\bar z)\Big[3\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - 2\mathcal{R}^{(2)}_{(3)}(z,\bar z|3) - 3\mathcal{R}^{(1)}_{(2)}(z,\bar z|2)\Big]
\end{aligned}
\tag{6.31}
$$

The general ansatz for the conformally covariant change of ghosts which does lead to the cancellation of the structure functions of the sub-levels ($s = 2, 3$) in the above variation (6.31) is given by

$$
\begin{aligned}
\mathcal{C}^{(2)}(z,\bar z) &= \widetilde{\mathcal{C}}^{(2)}(z,\bar z) + H_{(0)}(z,\bar z)\partial\mathcal{C}^{(3)}(z,\bar z) + H_{(1)}(z,\bar z)\mathcal{C}^{(3)}(z,\bar z),\\
\mathcal{C}^{(1)}(z,\bar z) &= \widetilde{\mathcal{C}}^{(1)}(z,\bar z) + F_{(0)}(z,\bar z)\partial\widetilde{\mathcal{C}}^{(2)}(z,\bar z) + F_{(1)}(z,\bar z)\widetilde{\mathcal{C}}^{(2)}(z,\bar z) + L_{(0)}(z,\bar z)\partial_{(2)}\mathcal{C}^{(3)}(z,\bar z)\\
&\quad + L_{(1)}(z,\bar z)\partial\mathcal{C}^{(3)}(z,\bar z) + L_{(2)}(z,\bar z)\mathcal{C}^{(3)}(z,\bar z).
\end{aligned}
\tag{6.32}
$$

Cancellation of the sub-levels in (6.31) gives

$$
\begin{aligned}
&H_{(0)}(z,\bar z) = -\tfrac{1}{2}, \quad F_{(0)}(z,\bar z) = -1, \quad H_{(1)}(z,\bar z) = \mathcal{R}^{(2)}_{(3)}(z,\bar z|3) - \tfrac{3}{4}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\\
&F_{(1)}(z,\bar z) = \mathcal{R}^{(1)}_{(2)}(z,\bar z|2) - \tfrac{1}{2}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4), \quad L_{(1)}(z,\bar z) = \tfrac{1}{4}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - \tfrac{1}{2}\mathcal{R}^{(1)}_{(2)}(z,\bar z|2) - \partial L_{(0)}(z,\bar z),
\end{aligned}
$$

and the glueing rules for $\mathcal{C}^{(1)}$ infers $L_{(0)}(z,\bar z) = \frac{1}{5}$. Plugging these results into (6.32) and inverting (6.30) the three $\mathcal{K}$ ghosts of the level $s = 4$ are re-expressed as

$$
\begin{aligned}
\mathcal{K}^{(3)}(z,\bar z|4) &= \mathcal{C}^{(3)}(z,\bar z)\\
\mathcal{K}^{(2)}(z,\bar z|4) &= \widetilde{\mathcal{C}}^{(2)}(z,\bar z) - \tfrac{1}{2}\partial\mathcal{C}^{(3)}(z,\bar z) - \tfrac{3}{4}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\mathcal{C}^{(3)}(z,\bar z)\\
\mathcal{K}^{(1)}(z,\bar z|4) &= \widetilde{\mathcal{C}}^{(1)}(z,\bar z) - \partial\widetilde{\mathcal{C}}^{(2)}(z,\bar z) - \tfrac{1}{2}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\widetilde{\mathcal{C}}^{(2)}(z,\bar z) + \tfrac{1}{5}\partial_{(2)}\mathcal{C}^{(3)}(z,\bar z)\\
&\quad + \tfrac{1}{4}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\partial\mathcal{C}^{(3)}(z,\bar z) + \mathcal{C}^{(3)}(z,\bar z)\Big(L_{(2)}(z,\bar z) + \tfrac{3}{4}\mathcal{R}^{(1)}_{(2)}(z,\bar z|2)\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - \mathcal{R}^{(1)}_{(3)}(z,\bar z|2)\Big).
\end{aligned}
\tag{6.33}
$$

Since there are various possibilities to cancel the sub-levels, the remaining function coefficient $L_{(2)}(z,\bar z)$ must be determined with the help of $\delta_{\mathcal{W}_4}\widetilde{\mathcal{C}}^{(2)}$ computed from the second equation of (6.33) and the known variations at the level $s = 4$ for $\mathcal{K}^{(3)}(z,\bar z|4)$, $\mathcal{K}^{(2)}(z,\bar z|4)$ and $\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)$ according to the general theory. This step will secure the nilpotency $\delta^2_{\mathcal{W}_4} = 0$. After lengthy computations performed with the help of `Mathematica`, one ends with

$$
\begin{aligned}
L_{(2)}(z,\bar z) &= \tfrac{12}{25}\partial\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - \tfrac{3}{25}\big(\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\big)^2 - \tfrac{41}{50}\mathcal{R}^{(2)}_{(4)}(z,\bar z|4)\\
&\quad - \tfrac{3}{4}\mathcal{R}^{(1)}_{(2)}(z,\bar z|2)\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) + \mathcal{R}^{(1)}_{(3)}(z,\bar z|2)\ ,
\end{aligned}
$$

which once substituted yields

$$
\begin{aligned}
\mathcal{K}^{(1)}(z,\bar z|4) &= \widetilde{\mathcal{C}}^{(1)}(z,\bar z) - \partial\widetilde{\mathcal{C}}^{(2)}(z,\bar z) - \tfrac{1}{2}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\widetilde{\mathcal{C}}^{(2)}(z,\bar z) + \tfrac{1}{5}\partial_{(2)}\mathcal{C}^{(3)}(z,\bar z)\\
&\quad + \tfrac{1}{4}\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\partial\mathcal{C}^{(3)}(z,\bar z) + \mathcal{C}^{(3)}(z,\bar z)\Big(\tfrac{12}{25}\big(\partial\mathcal{R}^{(3)}_{(4)}(z,\bar z|4) - \tfrac{1}{4}\big(\mathcal{R}^{(3)}_{(4)}(z,\bar z|4)\big)^2\big) - \tfrac{41}{50}\mathcal{R}^{(2)}_{(4)}(z,\bar z|4)\Big).
\end{aligned}
\tag{6.34}
$$

This shows that the system (6.11) can be rewritten in terms of the structure functions of the level $s = 4$ only thanks to the redefinition (6.32) of the tensorial ghosts coming from the sub-levels. Recall that these redefinitions are required for re-absorbing the structure functions of the sub-levels. The change of generators (6.33) also confirms the general ansatz (6.17) given in [15]. The variation (6.31) then reduces to

$$
\begin{aligned}
\delta_{\mathcal{W}_4}\mathcal{C}^{(3)}(z,\bar z) &= \widetilde{\mathcal{C}}^{(1)}(z,\bar z)\partial\mathcal{C}^{(3)}(z,\bar z) + 3\,\mathcal{C}^{(3)}(z,\bar z)\partial\widetilde{\mathcal{C}}^{(1)}(z,\bar z) + 2\,\widetilde{\mathcal{C}}^{(2)}(z,\bar z)\partial\widetilde{\mathcal{C}}^{(2)}(z,\bar z)\\
&\quad + \tfrac{1}{10}\Big(\mathcal{C}^{(3)}(z,\bar z)\partial_{(3)}\mathcal{C}^{(3)}(z,\bar z) - 2\,\partial\mathcal{C}^{(3)}(z,\bar z)\partial_{(2)}\mathcal{C}^{(3)}(z,\bar z) + 14\,\mathcal{T}_{(2)}(z,\bar z|4)\mathcal{C}^{(3)}(z,\bar z)\partial\mathcal{C}^{(3)}(z,\bar z)\Big)
\end{aligned}
\tag{6.35}
$$

from which emerges the projective connection $\mathcal{T}_{(2)}(z,\bar z|4)$ associated to the level $s = 4$,

$$
\mathcal{T}_{(2)}(z,\bar z|4) := \Big(\tfrac{3}{10}\partial\mathcal{R}^{(3)}_{(4)} - \tfrac{3}{40}\big(\mathcal{R}^{(3)}_{(4)}\big)^2 - \tfrac{1}{5}\mathcal{R}^{(2)}_{(4)}\Big)(z,\bar z|4) = \tfrac{1}{5}\,a^{(4)}_{(2)}(z,\bar z)
\tag{6.36}
$$

where $a^{(4)}_{(2)}$ given by (3.37) carries the projective connection property as it was already checked by using the general glueing rules (3.36). The variation $\delta_{\mathcal{W}_4}\widetilde{\mathcal{C}}^{(2)}$ is then computed to be

$$\begin{aligned}\delta_{\mathcal{W}_4}\widetilde{\mathcal{C}}^{(2)}(z,\overline{z}) &= \widetilde{\mathcal{C}}^{(1)}(z,\overline{z})\partial\widetilde{\mathcal{C}}^{(2)}(z,\overline{z}) + 2\,\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\partial\widetilde{\mathcal{C}}^{(1)}(z,\overline{z}) - \tfrac{3}{32}\,\mathcal{C}^{(3)}(z,\overline{z})\partial\mathcal{C}^{(3)}(z,\overline{z})W_{(3)}(z,\overline{z}|4)\\ &- \tfrac{1}{10}\Big(\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\partial_{(3)}\mathcal{C}^{(3)}(z,\overline{z}) - 3\,\partial\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\partial_{(2)}\mathcal{C}^{(3)}(z,\overline{z}) + 5\,\partial_{(2)}\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\partial\mathcal{C}^{(3)}(z,\overline{z})\\ &- 5\,\partial_{(3)}\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\mathcal{C}^{(3)}(z,\overline{z}) + \big(18\,\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\partial\mathcal{C}^{(3)}(z,\overline{z}) - 34\,\partial\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\mathcal{C}^{(3)}(z,\overline{z})\big)\mathcal{T}_{(2)}(z,\overline{z}|4)\\ &- 7\,\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\mathcal{C}^{(3)}(z,\overline{z})\partial\mathcal{T}_{(2)}(z,\overline{z}|4)\Big)\end{aligned}\tag{6.37}$$

where one also gets the conformally covariant spin three $\mathcal{W}$-current associated to the top level $s=4$,

$$W_{(3)}(z,\overline{z}|4) := \Big(8\partial\mathcal{R}^{(2)}_{(4)} - 4\partial_{(2)}\mathcal{R}^{(3)}_{(4)} + 3\partial\big(\mathcal{R}^{(3)}_{(4)}\big)^2 - 4\mathcal{R}^{(2)}_{(4)}\mathcal{R}^{(3)}_{(4)} - 8\mathcal{R}^{(1)}_{(4)}\Big)(z,\overline{z}|4).\tag{6.38}$$

Going on through the computation of the variation with the help of the third equation in (6.33) and the known variations at the level $s=4$ for $\mathcal{C}^{(3)}$, $\widetilde{\mathcal{C}}^{(2)}$, $\mathcal{R}^{(3)}_{(4)}(z,\overline{z}|4)$ and $\mathcal{R}^{(2)}_{(4)}(z,\overline{z}|4)$, and after redefining $\mathcal{C}^{(3)} := -320\,\widetilde{\mathcal{C}}^{(3)}$ by a numerical factor for later convenience, one gets,

$$\begin{aligned}\delta_{\mathcal{W}_4}\widetilde{\mathcal{C}}^{(1)}(z,\overline{z}) &= (\widetilde{\mathcal{C}}^{(1)}\partial\widetilde{\mathcal{C}}^{(1)})(z,\overline{z}) + \tfrac{3}{5}\Big(\partial\widetilde{\mathcal{C}}^{(2)}\partial_{(2)}\widetilde{\mathcal{C}}^{(2)} - \tfrac{2}{3}\widetilde{\mathcal{C}}^{(2)}\partial_{(3)}\widetilde{\mathcal{C}}^{(2)} - \tfrac{16}{3}\widetilde{\mathcal{C}}^{(2)}\partial\widetilde{\mathcal{C}}^{(2)}\mathcal{T}_{(2)}(z,\overline{z}|4)\Big)(z,\overline{z})\\ &+\Big(20\,\widetilde{\mathcal{C}}^{(2)}\partial\widetilde{\mathcal{C}}^{(3)} - \tfrac{108}{5}\partial\widetilde{\mathcal{C}}^{(2)}\widetilde{\mathcal{C}}^{(3)}\Big)(z,\overline{z})W_{(3)}(z,\overline{z}|4) + \tfrac{28}{5}\widetilde{\mathcal{C}}^{(2)}(z,\overline{z})\widetilde{\mathcal{C}}^{(3)}(z,\overline{z})\partial W_{(3)}(z,\overline{z}|4)\\ &+1024\Big[3\,\widetilde{\mathcal{C}}^{(3)}\partial_{(5)}\widetilde{\mathcal{C}}^{(3)} - 5\,\partial\widetilde{\mathcal{C}}^{(3)}\partial_{(4)}\widetilde{\mathcal{C}}^{(3)} + 6\,\partial_{(2)}\widetilde{\mathcal{C}}^{(3)}\partial_{(3)}\widetilde{\mathcal{C}}^{(3)} + 57\,\widetilde{\mathcal{C}}^{(3)}\partial_{(2)}\widetilde{\mathcal{C}}^{(3)}\partial\mathcal{T}_{(2)}(z,\overline{z}|4)\\ &+\Big(78\,\widetilde{\mathcal{C}}^{(3)}\partial_{(3)}\widetilde{\mathcal{C}}^{(3)} - 118\partial\widetilde{\mathcal{C}}^{(3)}\partial_{(2)}\widetilde{\mathcal{C}}^{(3)}\Big)\mathcal{T}_{(2)}(z,\overline{z}|4)\\ &+\widetilde{\mathcal{C}}^{(3)}\partial\widetilde{\mathcal{C}}^{(3)}\Big(57\partial_{(2)}\mathcal{T}_{(2)}(z,\overline{z}|4) + 432\big(\mathcal{T}_{(2)}(z,\overline{z}|4)\big)^2 - 14W_{(4)}(z,\overline{z}|4)\Big)\Big](z,\overline{z}),\end{aligned}\tag{6.39}$$

from where emerges a $\mathcal{W}$-current of spin 4 (a $(4,0)$-type conformally covariant differential) associated to the level $s=4$, as it can be checked by using (3.36),

$$\begin{aligned}800\,W_{(4)}(z,\overline{z}|4) = \Big(&144\,\big(\mathcal{R}^{(2)}_{(4)}\big)^2 + 400\,\mathcal{R}^{(1)}_{(4)}\mathcal{R}^{(3)}_{(4)} + 208\,\mathcal{R}^{(2)}_{(4)}\big(\mathcal{R}^{(3)}_{(4)}\big)^2 + 39\big(\mathcal{R}^{(3)}_{(4)}\big)^4 - 800\,\partial\mathcal{R}^{(1)}_{(4)}\\ &- 400\,\mathcal{R}^{(3)}_{(4)}\partial\mathcal{R}^{(2)}_{(4)} - 432\,\mathcal{R}^{(2)}_{(4)}\partial\mathcal{R}^{(3)}_{(4)} - 104\,\partial\big(\mathcal{R}^{(3)}_{(4)}\big)^3 + 264\,\big(\partial\mathcal{R}^{(3)}_{(4)}\big)^2\\ &+ 320\,\partial_{(2)}\mathcal{R}^{(2)}_{(4)} + 240\,\mathcal{R}^{(3)}_{(4)}\partial_{(2)}\mathcal{R}^{(3)}_{(4)} - 80\,\partial_{(3)}\mathcal{R}^{(3)}_{(4)}\Big)(z,\overline{z}|4).\end{aligned}\tag{6.40}$$

Hence, the general conformally covariant differential operator (3.1) for $s=4$ expressed in terms of the three $\mathcal{W}_4$-currents is

$$\begin{aligned}L_4(z,\overline{z}) = \partial_{(4)} &+ 5\mathcal{T}_{(2)}(z,\overline{z}|4)\partial_{(2)} + 5\partial\mathcal{T}_{(2)}(z,\overline{z}|4)\partial + \tfrac{3}{2}\Big(\partial_{(2)}\mathcal{T}_{(2)}(z,\overline{z}|4) + \tfrac{3}{2}\big(\mathcal{T}_{(2)}(z,\overline{z}|4)\big)^2\Big)\\ &+ \tfrac{1}{8}\,W_{(3)}(z,\overline{z}|4)\partial + \tfrac{1}{16}\partial W_{(3)}(z,\overline{z}|4) - \tfrac{1}{2}W_{(4)}(z,\overline{z}|4),\end{aligned}\tag{6.41}$$

where the first line is the Bol operator of order 4, see e.g. [24], depending only on the projective connection $\mathcal{T}_{(2)}$.

All this BRS algebra is an explicit realization of the so-called principal $\mathcal{W}_4$-algebra for pure $\mathcal{W}_4$-gravity [23, 36]. Performing the rescaling

$$\widetilde{\mathcal{C}}^{(2)} \longrightarrow -8i\sqrt{5}\,\widetilde{\mathcal{C}}^{(2)}, \qquad W_{(3)}(z,\overline{z}|4) \longrightarrow \tfrac{i}{8\sqrt{5}}\,W_{(3)}(z,\overline{z}|4),$$

and dropping out both the $\widetilde{\ }$ for the tensorial ghosts and the explicit reference to the the level $s=4$, one gets the presentation as a full nilpotent BRS algebra for $\mathcal{W}_4$-algebra,

$$
\begin{aligned}
\mathcal{S}\,\mathcal{C}^{(1)} &= \mathcal{C}^{(1)}\partial\mathcal{C}^{(1)} - 192\Big(\partial\mathcal{C}^{(2)}\partial_{(2)}\mathcal{C}^{(2)} - \tfrac{2}{3}\mathcal{C}^{(2)}\partial_{(3)}\mathcal{C}^{(2)} - \tfrac{16}{3}\mathcal{T}\mathcal{C}^{(2)}\partial\mathcal{C}^{(2)}\Big)\\
&\quad + 256\Big(27\partial\mathcal{C}^{(2)}\mathcal{C}^{(3)}W_{(3)} - 25\mathcal{C}^{(2)}\partial\mathcal{C}^{(3)}W_{(3)} - 7\mathcal{C}^{(2)}\mathcal{C}^{(3)}\partial W_{(3)}\Big)\\
&\quad + 1024\Big(3\mathcal{C}^{(3)}\partial_{(5)}\mathcal{C}^{(3)} - 5\partial\mathcal{C}^{(3)}\partial_{(4)}\mathcal{C}^{(3)} + 6\partial_{(2)}\mathcal{C}^{(3)}\partial_{(3)}\mathcal{C}^{(3)} + 57\partial_{(2)}\mathcal{T}\mathcal{C}^{(3)}\partial\mathcal{C}^{(3)}\\
&\qquad + 57\partial\mathcal{T}\mathcal{C}^{(3)}\partial_{(2)}\mathcal{C}^{(3)} + (78\mathcal{C}^{(3)}\partial_{(3)}\mathcal{C}^{(3)} - 118\partial\mathcal{C}^{(3)}\partial_{(2)}\mathcal{C}^{(3)})\mathcal{T}\\
&\qquad - 14\mathcal{C}^{(3)}\partial\mathcal{C}^{(3)}W_{(4)} + 432\mathcal{C}^{(3)}\partial\mathcal{C}^{(3)}\mathcal{T}^2\Big)\\
\mathcal{S}\,\mathcal{C}^{(2)} &= \mathcal{C}^{(1)}\partial\mathcal{C}^{(2)} + 2\mathcal{C}^{(2)}\partial\mathcal{C}^{(1)} + 32\Big(\mathcal{C}^{(2)}\partial_{(3)}\mathcal{C}^{(3)} - 3\partial\mathcal{C}^{(2)}\partial_{(2)}\mathcal{C}^{(3)} + 5\partial_{(2)}\mathcal{C}^{(2)}\partial\mathcal{C}^{(3)} - 5\partial_{(3)}\mathcal{C}^{(2)}\mathcal{C}^{(3)}\\
&\qquad + 18\mathcal{C}^{(2)}\partial\mathcal{C}^{(3)}\mathcal{T} - 34\partial\mathcal{C}^{(2)}\mathcal{C}^{(3)}\mathcal{T} - 7\mathcal{C}^{(2)}\mathcal{C}^{(3)}\partial\mathcal{T}\Big) - 9600\,\mathcal{C}^{(3)}\partial\mathcal{C}^{(3)}W_{(3)}\\
\mathcal{S}\,\mathcal{C}^{(3)} &= \mathcal{C}^{(1)}\partial\mathcal{C}^{(3)} + 3\mathcal{C}^{(3)}\partial\mathcal{C}^{(1)} + 2\mathcal{C}^{(2)}\partial\mathcal{C}^{(2)}\\
&\quad - 32\Big(\mathcal{C}^{(3)}\partial_{(3)}\mathcal{C}^{(3)} - 2\partial\mathcal{C}^{(3)}\partial_{(2)}\mathcal{C}^{(3)} + 14\mathcal{T}\mathcal{C}^{(3)}\partial\mathcal{C}^{(3)}\Big)\\
\mathcal{S}\,\mathcal{T} &= \partial_{(3)}\mathcal{C}^{(1)} + 2\mathcal{T}\partial\mathcal{C}^{(1)} + \mathcal{C}^{(1)}\partial\mathcal{T} - 8\Big(2\mathcal{C}^{(2)}\partial W_{(3)} + 3W_{(3)}\partial\mathcal{C}^{(2)}\Big)\\
&\quad + 32\Big(3\mathcal{C}^{(3)}\partial W_{(4)} + 4W_{(4)}\partial\mathcal{C}^{(3)}\Big) \qquad (6.42)\\
\mathcal{S}\,W_{(3)} &= \mathcal{C}^{(1)}\partial W_{(3)} + 3\,W_{(3)}\partial\mathcal{C}^{(1)} - 8\Big(\partial_{(5)}\mathcal{C}^{(2)} + 2\mathcal{C}^{(2)}\partial_{(3)}\mathcal{T} + 10\mathcal{T}\partial_{(3)}\mathcal{C}^{(2)} + 15\partial\mathcal{T}\partial_{(2)}\mathcal{C}^{(2)}\\
&\qquad + 9\partial_{(2)}\mathcal{T}\partial\mathcal{C}^{(2)} + 16\mathcal{T}\partial\mathcal{T}\mathcal{C}^{(2)} + 16\mathcal{T}^2\partial\mathcal{C}^{(2)} + \mathcal{C}^{(2)}\partial W_{(4)} + 2W_{(4)}\partial\mathcal{C}^{(2)}\Big)\\
&\quad + 32\Big(5\mathcal{C}^{(3)}\partial_{(3)}W_{(3)} + 10\partial\mathcal{C}^{(3)}\partial_{(2)}W_{(3)} + 28\partial_{(2)}\mathcal{C}^{(3)}\partial W_{(3)} + 14\partial_{(3)}\mathcal{C}^{(3)}W_{(3)}\\
&\qquad + 34\mathcal{C}^{(3)}\mathcal{T}\partial W_{(3)} + 27\mathcal{C}^{(3)}W_{(3)}\partial\mathcal{T} + 52\partial\mathcal{C}^{(3)}\mathcal{T}W_{(3)}\Big)\\
\mathcal{S}\,\mathcal{W}_{(4)} &= \mathcal{C}^{(1)}\partial W_{(4)} + 4W_{(4)}\partial\mathcal{C}^{(1)} - 8\Big(\mathcal{C}^{(2)}\partial_{(3)}W_{(3)} + 6\partial\mathcal{C}^{(2)}\partial_{(2)}W_{(3)} + 14\partial_{(2)}\mathcal{C}^{(2)}\partial W_{(3)} + 14\partial_{(3)}\mathcal{C}^{(2)}W_{(3)}\\
&\qquad + 18\mathcal{C}^{(2)}\mathcal{T}\partial W_{(3)} + 25\mathcal{C}^{(2)}\partial\mathcal{T}W_{(3)} + 52\partial\mathcal{C}^{(2)}\mathcal{T}W_{(3)}\Big)\\
&\quad + 32\Big(\partial_{(7)}\mathcal{C}^{(3)} + 3\mathcal{C}^{(3)}\partial_{(5)}\mathcal{T} + 20\partial\mathcal{C}^{(3)}\partial_{(4)}\mathcal{T} + 56\partial_{(2)}\mathcal{C}^{(3)}\partial_{(3)}\mathcal{T} + 84\partial_{(3)}\mathcal{C}^{(3)}\partial_{(2)}\mathcal{T} + 70\partial_{(4)}\mathcal{C}^{(3)}\partial\mathcal{T}\\
&\qquad + 28\partial_{(5)}\mathcal{C}^{(3)}\mathcal{T} + \mathcal{C}^{(3)}(177\partial\mathcal{T}\partial_{(2)}\mathcal{T} + 78\mathcal{T}\partial_{(3)}\mathcal{T}) + \partial\mathcal{C}^{(3)}(352\mathcal{T}\partial_{(2)}\mathcal{T} + 295(\partial\mathcal{T})^2)\\
&\qquad + 588\partial_{(2)}\mathcal{C}^{(3)}\mathcal{T}\partial\mathcal{T} + 196\partial_{(3)}\mathcal{C}^{(3)}\mathcal{T}^2 + 432\mathcal{C}^{(3)}\mathcal{T}^2\partial\mathcal{T} + 288\partial\mathcal{C}^{(3)}\mathcal{T}^3\\
&\qquad + 75\mathcal{C}^{(3)}W_{(3)}\partial W_{(3)} + 75\partial\mathcal{C}^{(3)}(W_{(3)})^2 - \mathcal{C}^{(3)}\partial_{(3)}W_{(4)} - 5\partial\mathcal{C}^{(3)}\partial_{(2)}W_{(4)} - 9\partial_{(2)}\mathcal{C}^{(3)}\partial W_{(4)}\\
&\qquad - 6\partial_{(3)}\mathcal{C}^{(3)}W_{(4)} - 14\mathcal{C}^{(3)}\partial(\mathcal{T}W_{(4)}) - 28\partial\mathcal{C}^{(3)}\mathcal{T}W_{(4)}\Big)\,.
\end{aligned}
$$

Remind once more that there is a breaking term in the top ghost variation $\mathcal{S}C^{(3)}$ with respect to the symplectic variation, so that the mechanisms using the so-called θ-trick described in previous papers [11, 31] for the $\mathcal{W}_3$ case does not work in the $\mathcal{W}_4$ case. Let us remark that if one sets $\mathcal{C}^{(3)}=0$ and $W_{(4)}=0$ and performs the rescalings of the generators $\mathcal{C}^{(2)} \longrightarrow \frac{i\sqrt{3}}{24}\,\mathcal{C}^{(2)}$ and $W_{(3)} \longrightarrow -8i\sqrt{3}\,W_{(3)}$ in (6.42) then the $\mathcal{W}_3$-algebra (6.29) is recovered. This confirms the universal definition (6.6) of the tensorial ghosts as $\mathcal{C}^{(s-1)}(z,\bar{z}) = \mathcal{K}^{(s-1)}(z,\bar{z}|s)$ as the top ghost of each level s and also the interweaving of the algebras dictated by the successive DOR's.

6.3 Comparison with some previous work

The general ansatz (6.17) given in [15] and exemplified in (6.22) and (6.33), (6.34) for $s=3,4$ respectively, can be put into relation with some previous pioneer work [6, 5, 8, 10]. Indeed, [10] will be of particular interest. There "Beltrami differentials" emerging from a multi-time approach for KdV flows were related to

"Bilal-Fock-Kogan" generalized tensorial Beltrami coefficients [5] appearing in $\mathcal{W}$-gravity along the ideas of [6]. According to their contravariant behavior these various type of Beltrami deformation parameters can be used in order to recover our ansatz (6.17).

As said in the Introduction, working with either homogeneous or inhomogeneous coordinates seems to be a matter of taste. In our construction, the latter were preferred because they strengthen the role of the symmetry algebra.

If one considers the homogeneous solutions f of the s-th order conformally covariant linear equation (3.1), these solutions as $\big(\frac{1-s}{2}\big)$-conformal fields are equivalently subject to a DOR since the s-th order derivative can be expressed in terms of the lower order ones and the smooth coefficient of the operator L_s. Their variation under large chiral diffeomorphisms were computed in [18] to be

$$\delta_{\mathcal{W}_s} f(z,\overline{z}) = \sum_{\ell=0}^{s-1} \mathcal{M}^{(\ell)}(z,\overline{z}|s)\partial_{(\ell)} f(z,\overline{z}) \ . \tag{6.43}$$

This variation for homogeneous coordinates must be related to the variation (4.1) for the inhomogeneous coordinates. Indeed Eqs.(3.8), (3.27) allow to find a complete link between the ghosts $\mathcal{K}^{(m)}(z,\overline{z}|s)$ and $\mathcal{M}^{(\ell)}(z,\overline{z}|s)$,

$$\mathcal{K}^{(m)}(z,\overline{z}|s) = \sum_{\ell=m}^{s-1} \binom{\ell}{m} \mathcal{M}^{(\ell)}(z,\overline{z}|s)\mathcal{Q}_{(\ell-m)}(z,\overline{z}|s), \qquad m=1,\dots,s-1 \tag{6.44}$$

and gives a direct answer to a problem raised in [6] about the $\mathcal{W}$ deformations of the f functions via the KdV "multi-time" approach, providing a direct expression of the KdV hierarchy.

Inspired by [10], one can mimic the construction used for relating KdV flows and $\mathcal{W}$-diffeomorphisms according to the following dictionary

$$\delta \longleftrightarrow \overline{\partial}, \qquad \mathcal{M}^{(\ell)} \longleftrightarrow \mu_\ell, \quad \text{and} \quad \widetilde{\mathcal{C}}^{(k)} \longleftrightarrow \rho_k, \tag{6.45}$$

where the $\mathcal{M}$ play the role of the ghost parameters for KdV flows and $\widetilde{\mathcal{C}}$[1] those for the infinitesimal $\mathcal{W}$-diffeomorphisms.

One can conformally covariantize the variation (6.43) by introducing tensorial ghosts $\widetilde{\mathcal{C}}$ which serve to filtrate the variation by their conformal weight according to

$$\delta_{\mathcal{W}_s} f = \sum_{k=1}^{s-1} \mathscr{B}_{(k)}(\widetilde{\mathcal{C}}^{(k)}, a^{(s)}_{(2)}) f \tag{6.46}$$

where the $\mathscr{B}_{(k)}$ are the conformally covariant differential operators constructed in [10] mapping $\big(\frac{1-s}{2}\big)$-conformal fields into themselves. The coefficient function $a^{(s)}_{(2)}$ has a prominent role since it is related to a projective connection (see (3.37)) and controls the Möbius transformations. For $a^{(s)}_{(2)} \equiv 0$ [2] one recalls that [10]

$$\mathscr{B}_{(k)}(\widetilde{\mathcal{C}}^{(k)}, a^{(s)}_{(2)} \equiv 0) = \sum_{j=k}^{s-1} \gamma^{(k-j)}_{(k)}[s]\big(\partial_{(mkj)}\widetilde{\mathcal{C}}^{(k)}\big)\partial_{(j)} \tag{6.47}$$

with $\gamma^{(0)}_{(k)}[s] = 1$ fixing the normalization between $\mathcal{M}$ and $\widetilde{\mathcal{C}}$. Comparison between the variations (6.43) and (6.46) yields

$$\begin{aligned} \mathcal{M}^{(0)}(z,\overline{z}|s) &= \sum_{k=1}^{s-1} \gamma^{(k)}_{(k)}[s]\partial_{(k)}\widetilde{\mathcal{C}}^{(k)}(z,\overline{z}) \\ \mathcal{M}^{(\ell)}(z,\overline{z}|s) &= \sum_{k=\ell}^{s-1} \gamma^{(k-\ell)}_{(k)}[s]\partial_{(k-\ell)}\widetilde{\mathcal{C}}^{(k)}(z,\overline{z}), \qquad \ell=1,\dots,s-1 \end{aligned} \tag{6.48}$$

[1] For the sake of consistency with the treated examples one uses the $\widetilde{\mathcal{C}}$ ghosts.

[2] Owing to (3.37) this implies a non trivial differential constraint on the structure functions and then a kind of group contraction.

where the numerical coefficients were given in [10]

$$\gamma^{(j)}_{(k)}[s] = (-1)^j \, \frac{\binom{s+j-k-1}{j}\binom{k}{j}}{\binom{2k}{j}}, \qquad \text{with } \gamma^{(0)}_{(k)}[s] = 1 \tag{6.49}$$

as solutions of the recursive equation

$$(j+1)(2k-j)\gamma^{(j+1)}_{(k)}[s] + (k-j)(s+j-k)\gamma^{(j)}_{(k)}[s] = 0$$

coming from the study of the covariance of (6.46) under projective holomorphic transformations.

Inserting (6.48) into (6.44) one gets at $a^{(s)}_{(2)} \equiv 0$

$$\mathcal{K}^{(m)}(z,\bar z|s) = \sum_{\ell=m}^{s-1} \binom{\ell}{m} \mathcal{Q}_{(\ell-m)}(z,\bar z|s) \sum_{k=\ell}^{s-1} \gamma^{(k-\ell)}_{(k)}[s] \partial_{(k-\ell)} \widetilde{\mathcal{C}}^{(k)}(z,\bar z), \qquad m = 1,\ldots,s-1 \tag{6.50}$$

The dependence in $a^{(s)}_{(2)}$ can be restored by studying the conformal covariance of (6.46) under an arbitrary holomorphic transformation. The change (6.50) corresponds to

$$\delta_{\mathcal{W}_s} Z(z,\bar z|s) = \sum_{k=1}^{s-1} \mathscr{D}_{(k)}(\widetilde{\mathcal{C}}^{(k)}, a^{(s)}_{(2)}) Z(z,\bar z|s) \tag{6.51}$$

an equivalent to (6.46). Thanks to filtration by the tensorial ghosts $\mathcal{C}^{(k)}$, for each k, the operator $\mathscr{D}_{(k)}(\widetilde{\mathcal{C}}^{(k)}, a^{(s)}_{(2)})$ acting on scalar fields has no constant term (by (4.1)) and must be a scalar under holomorphic transformations. They can be obtained by using the $\mathscr{B}$ operators computed in [10] without taking into account their constant terms since the inhomogeneous coordinates Z are used in the present paper. For the sake of completeness, one rewrites the first few of them

$$\begin{aligned}
\mathscr{D}_{(1)}(\widetilde{\mathcal{C}}^{(1)}, a^{(s)}_{(2)}) &= \widetilde{\mathcal{C}}^{(1)}\partial \\
\mathscr{D}_{(2)}(\widetilde{\mathcal{C}}^{(2)}, a^{(s)}_{(2)}) &= \widetilde{\mathcal{C}}^{(2)}\partial_{(2)} - \tfrac{s-2}{2}\partial\widetilde{\mathcal{C}}^{(2)}\partial \\
\mathscr{D}_{(3)}(\widetilde{\mathcal{C}}^{(3)}, a^{(s)}_{(2)}) &= \widetilde{\mathcal{C}}^{(3)}\partial_{(3)} - \tfrac{s-3}{2}\partial\widetilde{\mathcal{C}}^{(2)}\partial_{(2)} + \Big(\tfrac{(s-2)(s-3)}{10}\partial_{(2)}\widetilde{\mathcal{C}}^{(3)} + \tfrac{6(3s^2-7)}{5(s^3-s)}\widetilde{\mathcal{C}}^{(3)}a^{(s)}_{(2)}\Big)\partial
\end{aligned} \tag{6.52}$$

A direct confrontation of (6.50) (in which the $a^{(s)}_{(2)}$-dependence has been made explicit) with (6.22) and (6.33), (6.34) for respectively $s = 3,4$ gives a perfect accord upon using the recursion (3.27) and the definition (3.37).

The general ansatz (6.17) can be thus recovered with the help of existing results in the literature. But the linear decomposition (6.11) depending on the structure functions of all the possible sub-levels shows the origin of the tensorial ghosts as the highest conformal weighted parameter in each of the nested sub-algebras governed by the DOR filtration. According to the treated examples $\mathcal{W}_3$ and $\mathcal{W}_4$-algebras, the appropriate ghost parameters for the linear $\mathcal{W}$-diffeomorphisms are those constructed by redefining $\mathcal{C}^{(\ell)} \longrightarrow \widetilde{\mathcal{C}}^{(\ell)}$, for the intermediate DOR decompositions in order to re-absorb all the structure functions of the sub-levels. It is worthwhile to notice that the algorithm is performed in a conformally covariant manner and in the respect of the nilpotency of the $\mathcal{W}$-algebra.

7 Conclusion and perspective

Throughout the paper, we have considered conformal differential operators defined on a Riemann surface whose solutions are homogeneous coordinates of some complex projective space. The latter lead to the notion of Forsyth frames as projective coordinates. In this context, our main results are:

(i) linear differential order reductions (DOR), see Theorem 3.1, determine the structure functions of the large chiral symmetry algebra. These structure functions are the central objects of all our construction;

(ii) conformal differential operators can be explicitly constructed from the given structure functions entering the linear DOR;

(iii) the extension to the chiral truncated Taylor expansion of complex scalar fields of the usual infinitesimal chiral diffeomorphisms induces an algebraic framework, which, embedded into a B.R.S. setting, leads to another presentation of $\mathcal{W}$-algebras (Eq.(4.2)) written in terms of jet-ghosts.

(iv) due to physical considerations require to transform these ghosts from jets into tensors. Obviously, this change of generators is not unique.

In doing so, we have given a general solution (for any order s of the algebra), which put into the game all the truncation mechanisms of the Forsyth frames up to order s via a differential order reduction (DOR). The price to pay in keeping the entire generality of the solution is to carry the weight of the whole hierarchy of differential equations (with orders lower than s) which rule all the linear truncations. However, if one considers, for a given level, the general solution (6.14) as (physically) uncompleted, the removing of the role of the intermediate levels to the benefit of the standard $\mathcal{W}$-algebra presentation comes as a satisfying surprise. It has been shown that the cancellation holds in a rather tractable way for the lowest orders and can be related with some known computation [10]. However, the existence of non trivial redefinitions of the $\mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ ghosts leading to the re-absorption of the intermediate DOR decompositions, could be an very interesting problem. This gives a sharper indication on the nature of the tensorial ghosts $\widetilde{\mathcal{C}}$ associated to infinitesimal $\mathcal{W}$-diffeomorphisms. It is a close issue to the one concerning the relationship between the parameters of the KdV flows and those of infinitesimal $\mathcal{W}$-diffeomorphisms [10]. In particular, how the nested variations pertaining to the various sub-levels are finally disentangled to the benefit of the top level only, deserve to be better studied. All comes from both the conformal covariance (geometry and global meaning) and the nilpotency of the BRS operation (associative algebra of symmetry). This gives an algorithm similar to one obtained in [10], in which, conformal covariance governs the calculation as well.

(v) $\mathcal{W}$-currents are differential polynomials in the structure functions $\mathcal{R}(z,\overline{z}|s)$ only.

Further, a window on the so-called $\mathcal{W}$-gravity is open, once the BRS algebra is given, with the use of the algebraic trick given in [28, 11, 32] in order to incorporate the sources of the $\mathcal{W}$-currents. The relationship between the $\mathcal{W}$-diffeomorphism symmetry and the Beltrami deformation parameters for the complex geometry is given by

$$\overline{\partial} = \{\delta_{\mathcal{W}_s}, \frac{\partial}{\partial \overline{c}}\}, \qquad \rho^{(\ell)} = \frac{\partial \widetilde{\mathcal{C}}^{(\ell)}}{\partial \overline{c}} \ ,$$

where $\overline{c}$ is the true diffeomorphism ghost along the direction $\overline{\partial}$ and the $\rho^{(\ell)}$ are expected to be the sources for the $\mathcal{W}$-currents. This justifies (6.45) and allows to get the whole BRS algebra for $\mathcal{W}$-gravity directly from the BRS algebra for $\mathcal{W}$-algebra (e.g. (6.29) or (6.42)). In particular, this will be useful for a systematic study of $\mathcal{W}$-anomalies possibly arising at the quantum level.

As a final conclusion, we emphasize once more that, due to the non linearity of this type of symmetry algebra of large (chiral) diffeomorphisms, the technical intricacy is just a consequence of the reduction from jets to tensors for which non trivial explicit solutions have been obtained. The latter can be considered as a starting point for a more pleasant treatment, and a more suitable physical formulation for general $\mathcal{W}$-algebras and their relationship not only with linear algebraic differential equations [14, 6], but also with some kind of differential systems. For instance, one ought to expect that the Bershadsky $\mathcal{W}_3^{(2)}$-algebra be rather related to a conformally covariant system of coupled differential equations (with as unknowns f and g) of the form [12]

$$\begin{aligned}\big(\partial_{(2)} + a_{(1)}(z,\overline{z})\partial + a_{(2)}(z,\overline{z})\big)f(z,\overline{z}) + b(z,\overline{z})g(z,\overline{z}) \;&=\; 0\\ \big(\partial - \tfrac{1}{2}a_{(1)}(z,\overline{z})\big)g(z,\overline{z}) + B(z,\overline{z})f(z,\overline{z}) \;&=\; 0\ ,\end{aligned}$$

over a generic Riemann surface.

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